Answers to lecture problems – lectures 10…13

Lecture 10

Slide 1

Order is 2, equation is linear and homogeneous.

$$\frac{dN}{dt} = -\lambda N$$
$$\frac{dN}{N} = -\lambda dt$$
$$\ln N = -\lambda t + k$$
$$N = \exp\left[-\lambda t + k\right]$$
$$= \exp\left[k\right]\exp\left[-\lambda t\right]$$

Usually written $N = N_0 \exp\left[-\lambda t\right]$

Slide 3

Order 1, linear, inhomogeneous

Order 2, linear, homogeneous

Order 1, non-linear, inhomogeneous

Order 2, linear, homogeneous

Order 2, linear, homogeneous

Order 2, non-linear, inhomogeneous

Lecture 11

Slide 1

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$
$$m^2 - 4m + 4 = 0$$
$$(m-2)(m-2) = 0$$

Double root $m = 2$

$$y = A\exp\left[2x\right] + Bx\exp\left[2x\right]$$
$$\frac{dy}{dx} = 2A\exp\left[2x\right] + B\exp\left[2x\right] + 2Bx\exp\left[2x\right]$$

$y(1) = 1$ so $A\exp\left[2\right] + B\exp\left[2\right] = 1$

$\frac{dy(1)}{dx} = 1$ so $2A\exp\left[2\right] + 3B\exp\left[2\right] = 1$

Hence $\mathrm{B}\exp\left[2\right] = -1$ and $\mathrm{A}\exp\left[2\right] = 2$ so $\mathrm{B} = -\frac{1}{\exp\left[2\right]}$ and $\mathrm{A} = \frac{2}{\exp\left[2\right]}$.

$$y = (2-x)\exp\left[2x-2\right].$$

Lecture 12

Slide 1

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=x^2+5$$

Solution of auxiliary equation is

$$y_c=C_1\exp[-3x]+C_2\exp[x]$$

Trial solution

$$y_p=Ax^2+Bx+C$$

Equating coefficients gives

$$A=-\frac{1}{3},\ B=-\frac{4}{9}C \text{ and } =-\frac{59}{27}$$

$$y=y_c+y_p=C_1\exp[-3x]+C_2\exp[x]-\frac{x^2}{3}-\frac{4x}{9}-\frac{59}{27}$$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=5\exp[-3x]$$

Solution of auxiliary equation is (as above)

$$y_c=C_1\exp[-3x]+C_2\exp[x]$$

Trial solution

$$y_p=Ax\exp[-3x]$$

Equating coefficients gives

$$A=-\frac{5}{4}$$

$$y=y_c+y_p=C_1\exp[-3x]+C_2\exp[x]-\frac{5}{4}x\exp[-3x]$$

Lecture 13

Slide 1

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=5\exp[-3x]$$

Solution of auxiliary equation is (as for Lecture 12 above)

$$y_c=C_1\exp[-3x]+C_2\exp[x]$$

Trial solution

$$y_p=A\sin 3x+B\cos 3x$$

Equating coefficients gives

$$A=\frac{7}{30},\ B=-\frac{7}{15}$$

$$y=y_c+y_p=C_1\exp[-3x]+C_2\exp[x]+\frac{7}{30}\sin 3x-\frac{7}{15}\cos 3x.$$

Second problem is tackled in the lecture!

$$y=C\exp[Ax].$$