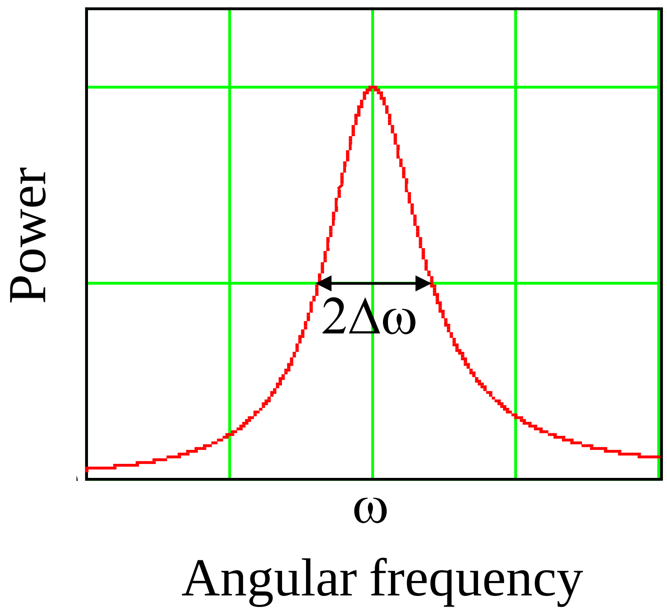
Lecture 21

- In this lecture we will look at:
  - Power and resonance.
  - The quality factor of a circuit.
  - Describing AC circuits using complex numbers.

- After this lecture, you should be able to answer the following questions:
- Describe how the power dissipated in a series LCR circuit varies with the frequency of the signal driving the circuit.
- How is the Q value of a circuit related to the width of the peak in the power spectrum at resonance?
- How would you ensure that a series LCR circuit had a large Q value?

Power and Resonance

- Average power maximum when $\cos\phi = 1$, i.e. $\phi = 0$, the resonance condition.
- Look at power as function of frequency:



- Full width (of peak at) half maximum (power) FWHM = $2\Delta\omega$.
- Define the quality factor (the sharpness of the resonance):

$$Q = \frac{\omega}{2\Delta\omega} \qquad [21.1]$$

- Now:

$$P_{rms} = I_{rms}^2 R = \frac{\mathcal{E}_{rms}^2 R}{R^2 + (\omega_d L - 1/\omega_d C)^2}$$

$$P_{peak} = \frac{\mathcal{E}_{rms}^2 R}{R^2} = \frac{\mathcal{E}_{rms}^2}{R}.$$

- To work out width, need ω_d at which power has decreased to $P_{peak}/2$:

$$\frac{P_{rms}}{P_{peak}} = \frac{\mathcal{E}_{rms}^2 R}{R^2 + (\omega_d L - 1/\omega_d C)^2} \times \frac{R}{\mathcal{E}_{rms}^2} = \frac{1}{2}.$$

Power and Resonance: Quality factor of Circuit

- From this we get:

$$\frac{R^2}{R^2 + (\omega_d L - 1/\omega_d C)^2} = \frac{1}{2}$$

$$\Rightarrow R^2 = (\omega_d L - 1/\omega_d C)^2$$

$$\text{or } \omega_d L - 1/\omega_d C = \pm R$$

- This can be written as a quadratic equation in ω_d:

$$L\omega_d^{\;2} \mp R\omega_d - 1/C = 0$$

$$\Rightarrow \omega_d = \frac{\pm R \pm \sqrt{R^2 + 4L/C}}{2L}.$$

- Only +ive solutions of this make sense (can't have –ive frequency!) so discard the two solutions with the second minus sign.

- The remaining +ive solutions must represent the $\omega + \Delta\omega$ and $\omega - \Delta\omega$.
- Subtracting these gives:

$$2\Delta\omega = \frac{+R + \sqrt{R^2 + 4L/C}}{2L} - \frac{-R + \sqrt{R^2 + 4L/C}}{2L}$$

$$= R/L.$$

- Hence:

$$Q = \frac{\omega}{2\Delta\omega} = \frac{\omega}{R/L}, \text{ so } Q = \frac{\omega L}{R} \qquad [21.2]$$

- At resonance, $\omega L = 1/\omega C$, so we can also write this:

$$Q = \frac{1/\omega C}{R} = \frac{1}{\omega C}\frac{1}{R} \qquad [21.3]$$

- We see small R relative to X_L or X_C means large Q or sharp resonance.

Quality Factor of Circuit and Energy Loss

- There is a second definition of the quality factor of a circuit:

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per cycle}} = \frac{\text{energy stored}}{\text{energy lost per radian}}.$$

- The energy stored (e.g. when it is all in the inductance) is:

$$E_{\text{stored}} = \tfrac{1}{2} L I^2.$$

- The energy lost per cycle is:

$$E_{\text{lost}} = P_{\text{avg}} T = \frac{I^2 R}{2} T = \frac{I^2 R}{2} \frac{2\pi}{\omega}.$$

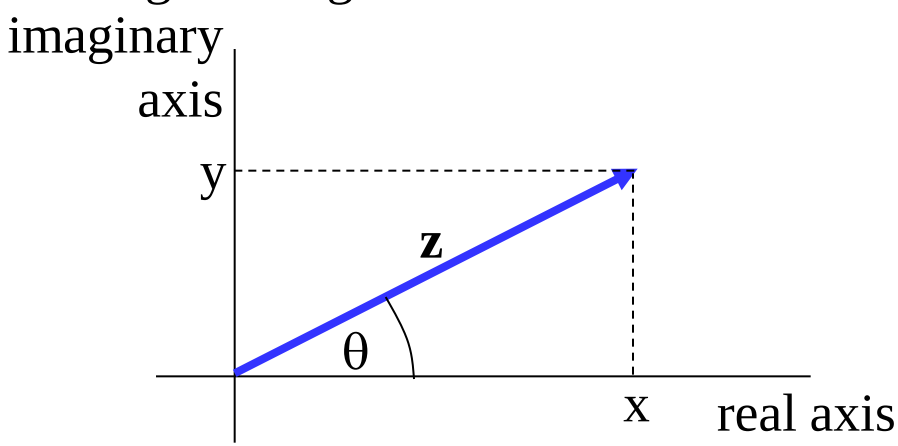
- Hence:

$$Q = 2\pi \frac{\tfrac{1}{2} L I^2}{I^2 R / 2 \times 2\pi / \omega}, \text{ or } Q = \frac{\omega L}{R}.$$

- The two definitions lead to the same mathematical expression.
- A circuit with a large Q value is thus one for which the resonance is sharp, but also one for which the energy lost per cycle is a small proportion of the stored energy.
- This is the end of the course!

Complex Numbers

- An alternative formalism for describing AC circuits is offered by complex numbers.
- A complex number can be written:
  $\mathbf{z} = x + jy.$
- The real part of **z** is x, that is:
  $\mathrm{Re}(\mathbf{z}) = x.$
- The imaginary part of **z** is y, that is:
  $\mathrm{Im}(\mathbf{z}) = y.$
- The quantity j is defined by:
  $j^2 = -1 \text{ or } j = \sqrt{-1}.$
- A point in the complex plane can be used to represent **z** (in the Argand diagram).
- Argand diagram:



- De Moivre's theorem states:

$$\exp[j\theta] = \cos\theta + j\sin\theta$$

$$\Rightarrow \cos\theta = \mathrm{Re}\left(\exp[j\theta]\right)$$

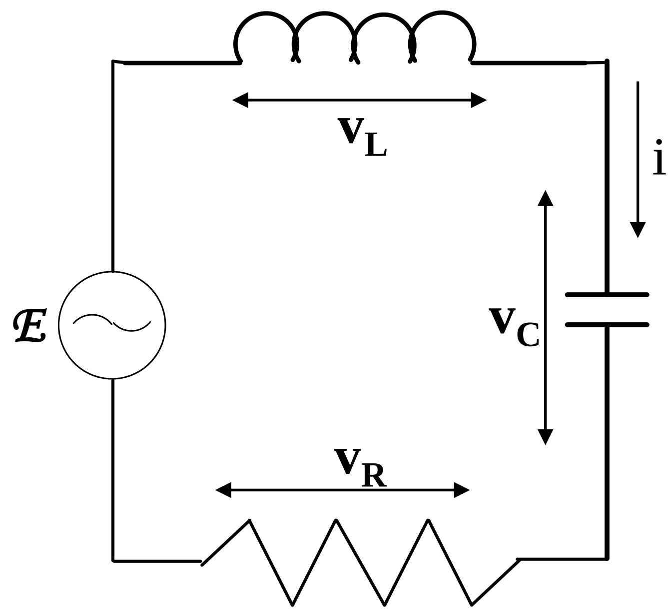
$$\text{and } \sin\theta = \mathrm{Im}\left(\exp[j\theta]\right).$$

- Hence $\mathbf{z} = z\exp[j\theta]$, where:

$$z = |\mathbf{z}| = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left[\frac{y}{x}\right].$$

Complex Numbers and AC Circuits

- Can represent quantities in AC circuits, e.g. emf $\mathcal{E} = \mathcal{E}_m \sin(\omega_d t)$, using:
  - Phasors, vector length E_m rotating at angular frequency ω.
  - Complex number, $\boldsymbol{\mathcal{E}} = \mathcal{E}_m \exp[j\omega_d t]$.
- Choosing the imaginary part of $\boldsymbol{\mathcal{E}}$ would give for the emf: $\mathcal{E} = \mathrm{Im}(\boldsymbol{\mathcal{E}}) = \mathcal{E}_m \sin(\omega_d t)$.
- Choosing the real part gives: $\mathcal{E} = \mathrm{Re}(\boldsymbol{\mathcal{E}}) = \mathcal{E}_m \cos(\omega_d t)$.
- Usually, the real part is used to represent the emf driving a circuit: in the rest of this lecture, $\mathcal{E} = \mathcal{E}_m \cos(\omega_d t)$.
- Complex numbers allow a compact and convenient description of AC circuits.
- Consider our series LCR circuit:



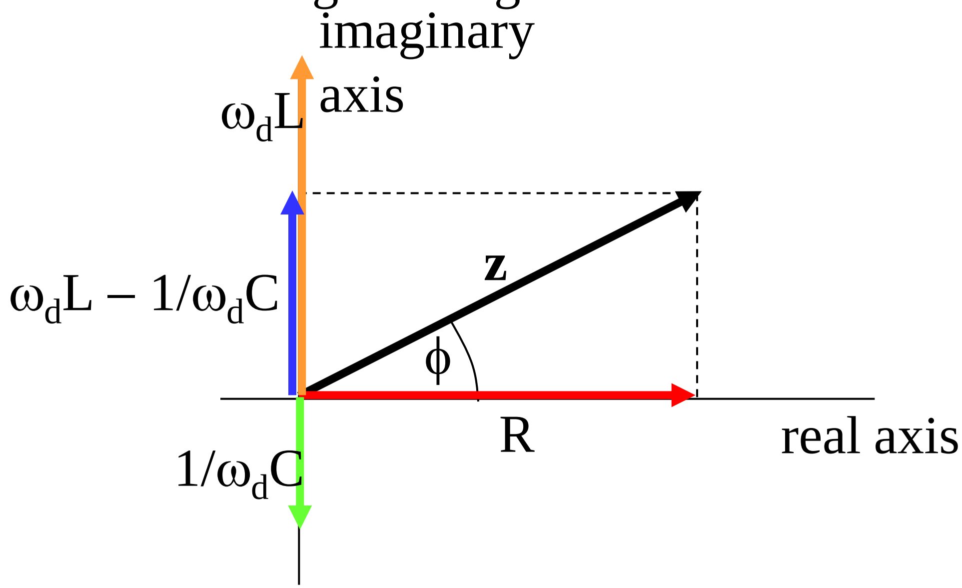
- Kirchoff's loop rule tells us: $\boldsymbol{\mathcal{E}} = \mathbf{v}_R + \mathbf{v}_C + \mathbf{v}_L$.

Complex Numbers and AC Circuits

- Using the familiar expressions for the voltage across the capacitance and the inductance:
$$\boldsymbol{\mathcal{E}} - L\frac{d\mathbf{i}}{dt} - \frac{1}{C}\int \mathbf{i}\,dt = \mathbf{i}R.$$
- We are representing **i** as a complex number as it must also be sinusoidal.
- We must also allow the phase of i to be different to that of v, so we write:
$$\mathbf{i} = I\exp\left[j(\omega_d t - \phi)\right].$$
- Differentiating and integrating **i** we have:
$$\frac{d\mathbf{i}}{dt} = j\omega_d I\exp\left[j(\omega_d t - \phi)\right] \text{ and}$$
$$\int \mathbf{i}\,dt = \frac{1}{j\omega_d} I\exp\left[j(\omega_d t - \phi)\right].$$
- Substituting these expressions into our integro-differential equation:
$$\mathcal{E}_m \exp\left[j\omega_d t\right] - j\omega_d L I\exp\left[j(\omega_d t - \phi)\right] - \frac{1}{j\omega_d C} I\exp\left[j(\omega_d t - \phi)\right] = RI\exp\left[j(\omega_d t - \phi)\right].$$
- Rearranging gives:
$$\mathcal{E}_m \exp\left[j\omega_d t\right] = I\exp\left[j(\omega_d t - \phi)\right] \times \left(R + j\omega_d L + \frac{1}{j\omega_d C}\right).$$
- Using $j^2 = -1$:
$$\mathcal{E}_m \exp\left[j\omega_d t\right] = I\exp\left[j(\omega_d t - \phi)\right] \times \left(R + j\left(\omega_d L - \frac{1}{\omega_d C}\right)\right).$$

Complex Numbers and AC Circuits

- We see the equation
  $$\mathcal{E}_m \exp\left[j\omega_d t\right] = I\exp\left[j(\omega_d t - \phi)\right]\times \left(R + j\left(\omega_d L - \frac{1}{\omega_d C}\right)\right),$$
  has the form $\boldsymbol{\mathcal{E}} = \mathbf{i}\,\mathbf{Z}$, where $\mathbf{Z}$ is the quantity:
  $$\mathbf{Z} = R + j\left(\omega_d L - \frac{1}{\omega_d C}\right).$$
- $\mathbf{Z}$, the complex impedance, can be expressed in the alternative form:
  $$\mathbf{Z} = Z\exp[j\phi], \text{ where:}$$
  $$Z = \sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2} \text{ and}$$
  $$\phi = \tan^{-1}\left(\frac{\omega_d L - 1/\omega_d C}{R}\right).$$
- In the Argand diagram:



- See similarity to phasor approach.
- Resistive component of Z same phase as $\mathcal{E}$, inductive component leads and capacitive component lags $\mathcal{E}$ by $\pi/2$.
- Voltages associated with these components behave similarly.

Complex Impedance and Circuits with only R or L

- For resistor:

  $$\mathbf{Z} = R + j0 \Rightarrow Z = \sqrt{R^2 + 0^2} = R$$

  and $\phi = \tan^{-1}(0/R) = 0.$

- From $\boldsymbol{\mathcal{E}} = \mathbf{i}\,\mathbf{Z}$ can calculate complex current:

  $$\mathbf{i} = \frac{\boldsymbol{\mathcal{E}}}{\mathbf{Z}} = \frac{\mathcal{E}_m \exp[j\omega_d t]}{Z\exp[j0]} = \frac{\mathcal{E}_m}{R}\exp[j\omega_d t].$$

- The current through the circuit is given by the real part of this:

  $$i = \mathrm{Re}(\mathbf{i}) = \frac{\mathcal{E}_m}{R}\cos(\omega_d t).$$

- This is in phase with the emf, remember here $\mathcal{E} = \mathcal{E}_m \cos(\omega_d t)$, and has amplitude $\mathcal{E}_m/R$ as we expect.

- For inductor:

  $$\mathbf{Z} = 0 + j\omega_d L \Rightarrow Z = \sqrt{0^2 + (\omega_d L)^2} = \omega_d L$$

  and $\phi = \tan^{-1}(\omega_d L/0) = \pi/2.$

- Current:

  $$\mathbf{i} = \frac{\boldsymbol{\mathcal{E}}}{\mathbf{Z}} = \frac{\mathcal{E}_m \exp[j\omega_d t]}{Z\exp[j\pi/2]} = \frac{\mathcal{E}_m}{\omega_d L}\exp[j(\omega_d t - \pi/2)].$$

- Take real part of this:

  $$i = \mathrm{Re}(\mathbf{i}) = \frac{\mathcal{E}_m}{\omega_d L}\cos(\omega_d t - \pi/2).$$

- Again, result is as expected: current lags behind emf by $\pi/2$ and has amplitude $\mathcal{E}_m/\omega_d L$.

Complex Impedance with only C: General Circuits

- For capacitor:

$$\mathbf{Z} = 0 - j/\omega_d L$$

$$\Rightarrow Z = \sqrt{0^2 + (1/\omega_d C)^2} = 1/(\omega_d C)^2$$

$$\text{and } \phi = \tan^{-1}\left(\frac{-1/\omega_d C}{0}\right) = -\frac{\pi}{2}.$$

- Current: $\mathbf{i} = \frac{\boldsymbol{\mathcal{E}}}{\mathbf{Z}} = \frac{\mathcal{E}_m \exp[j\omega_d t]}{Z \exp[-j\pi/2]}$

$$= \frac{\mathcal{E}_m}{1/\omega_d C}\exp[j(\omega_d t + \pi/2)].$$

- Take real part of this:

$$i = \mathrm{Re}(\mathbf{i}) = \frac{\mathcal{E}_m}{1/\omega_d C}\cos(\omega_d t + \pi/2).$$

- As expected: current leads emf by $\pi/2$ and has amplitude $\mathcal{E}_m/(1/\omega_d C)$.

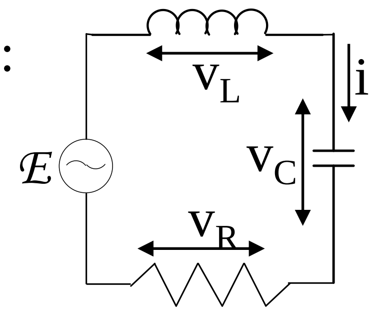
- For general AC circuit:
- Determine impedance of each component.
- Combine to give total impedance:
  - Series $\mathbf{Z}_{total} = \sum_i \mathbf{Z}_i$.
  - Parallel $\frac{1}{\mathbf{Z}_{total}} = \sum_i \frac{1}{\mathbf{Z}_i}$.
- Complex current from $\mathbf{i} = \boldsymbol{\mathcal{E}}/\mathbf{Z}_{total}$.
- Amplitudes and phases from:

$$\frac{\mathcal{E}_m}{I} = |\mathbf{Z}_{total}| \text{ and } \tan^{-1}\left(\frac{\mathrm{Im}(\mathbf{Z}_{total})}{\mathrm{Re}(\mathbf{Z}_{total})}\right).$$

- Current from $\mathrm{Re}(\mathbf{i})$.

Complex Impedance and Series LCR Circuit

- Complex impedances:
  - $\mathbf{Z}_R = R.$
  - $\mathbf{Z}_L = j\omega_d L.$
  - $\mathbf{Z}_C = -j/\omega_d C.$



- Add in series: $\mathbf{Z}_{total} = \mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C$

$$= R + j\omega_d L - j/\omega_d C$$

$$= R + j(\omega_d L - 1/\omega_d C).$$

- Hence:
  - $Z_{total} = |\mathbf{Z}_{total}| = \sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}.$
  - $\phi = \tan^{-1}\left(\frac{\mathrm{Im}(\mathbf{Z}_{total})}{\mathrm{Re}(\mathbf{Z}_{total})}\right)$

$$= \tan^{-1}\left(\frac{\omega_d L - 1/\omega_d C}{R}\right).$$

- Current:

$$i = \mathrm{Re}(\mathbf{i}) = \mathrm{Re}\left(\frac{\boldsymbol{\mathcal{E}}}{\mathbf{Z}_{total}}\right)$$

$$= \mathrm{Re}\left(\frac{\mathcal{E}_m \exp[j\omega_d t]}{Z_{total}\exp[j\phi]}\right)$$

$$= \mathrm{Re}\left(\frac{\mathcal{E}_m}{Z_{total}}\exp[j(\omega_d t - \phi)]\right)$$

$$= \frac{\mathcal{E}_m}{Z_{total}}\cos(\omega_d t - \phi).$$