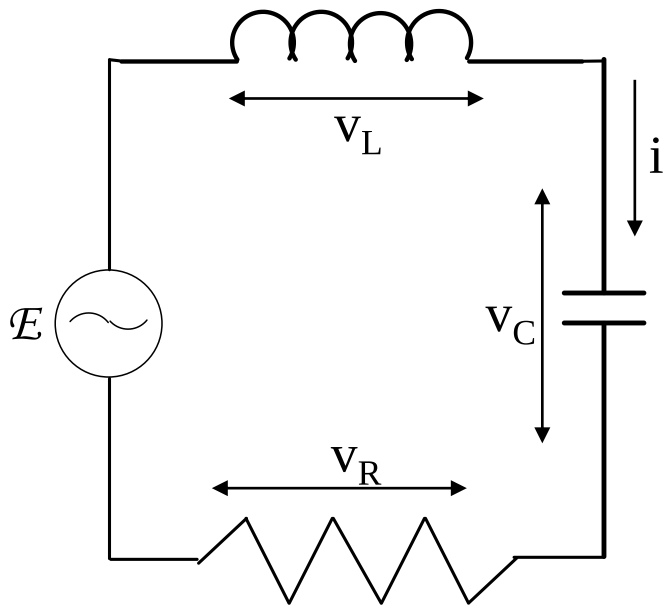
Lecture 20

- In this lecture we will look at:
  - Series LCR circuit.
  - Resonance.
  - Transients.
  - Power in AC circuits.

- After this lecture, you should be able to answer the following questions:
- What is the impedance of a resistance R, an inductance L and a capacitance C connected in series?
- Describe how the capacitive and inductive reactances change as the frequency of the sinusoidal signal driving a series LCR circuit increases from well below the circuit's resonant frequency to well above it.
- Explain the difference between the amplitude of an AC current and the rms value of the current. Why is the rms value a useful quantity?

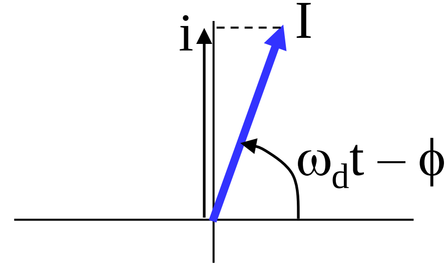
Series LCR Circuit

- Have looked at AC circuits containing L, C and R separately.
- What happens when all are present, connected in series?

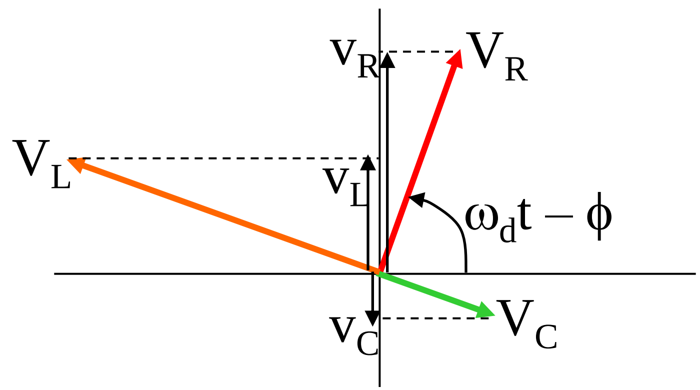


- Same current flows through all components.

- Draw phasors for circuit, first current, $i = I \sin(\omega_d t - \phi)$:

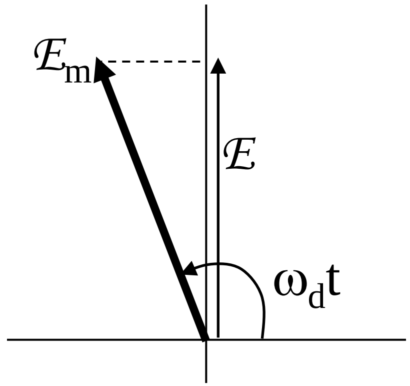


- Now phasors for v_R, v_C and v_L, with the phase relationships: i same as v_R; i leads v_C by $\pi/2$; i lags v_L by $\pi/2$.

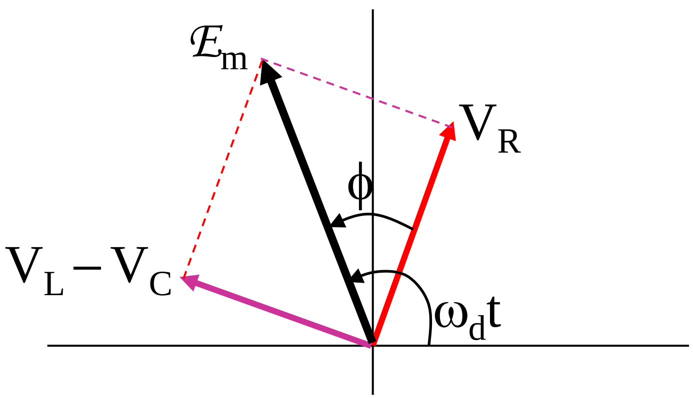


Series LCR Circuit

- Phasor for emf:



- From the loop rule $\mathcal{E} = v_R + v_C + v_L$.
- Hence sum of phasors for v_L, v_C and v_R must give phasor for emf:



- Phasors for v_L and v_C have opposite directions, so mag. of sum is $V_L - V_C$.
- From Pythagoras' theorem,
  $\mathcal{E}_m{}^2 = V_R{}^2 + (V_L - V_C)^2$.
- Remember $V_R = I\,R$, $V_C = I\,X_C$ and $V_L = I\,X_L$, so:
  $\mathcal{E}_m{}^2 = (IR)^2 + (IX_L - IX_C)^2$.
- Rearranging allows us to determine the amplitude of the current:

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathcal{E}_m}{Z} \qquad [20.1]$$

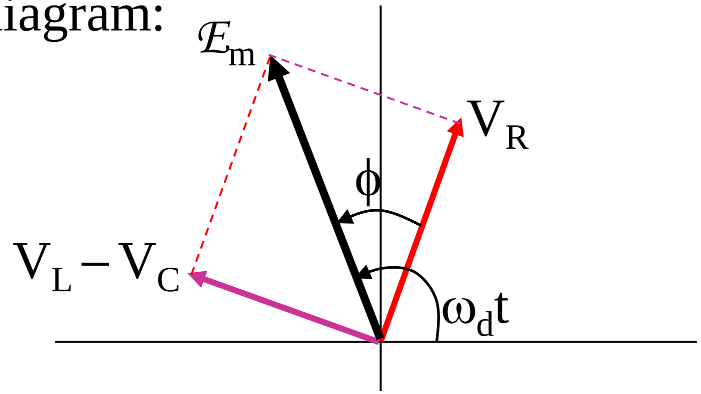
- Here we have defined the impedance:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{R^2 + \left(\omega_d L - 1/\omega_d C\right)^2} \qquad [20.2]$$

Series LCR Circuit

- What about the relative phase of the current and the voltage?
- Look again at the LCR phasor diagram:

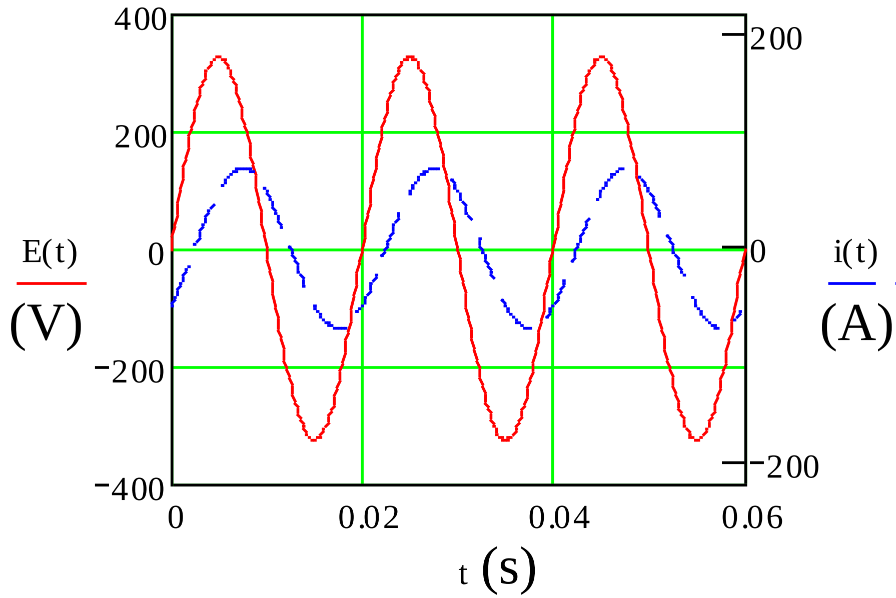


- We see $\tan\phi = \dfrac{V_L - V_C}{V_R}$

$$= \frac{IX_L - IX_C}{IR}$$

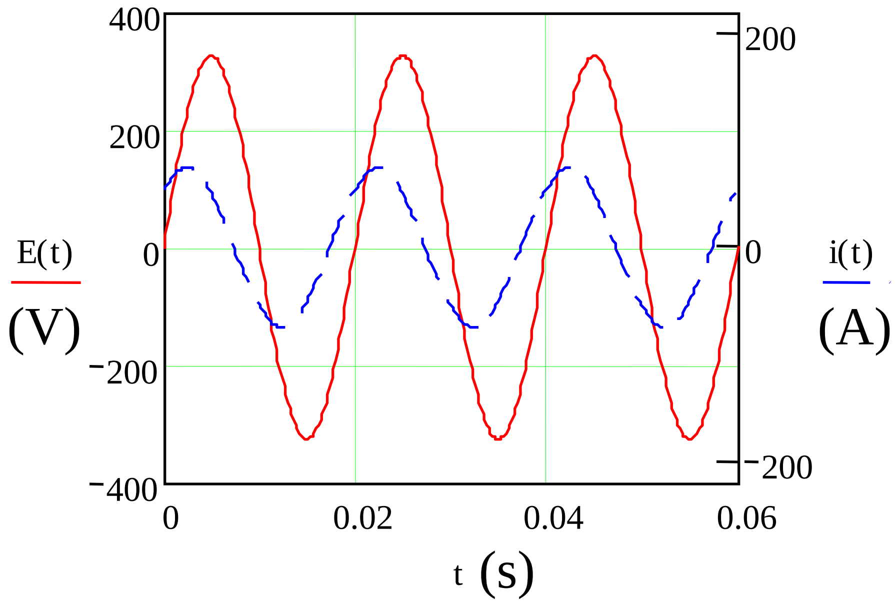
$$= \frac{X_L - X_C}{R} \qquad [20.3]$$

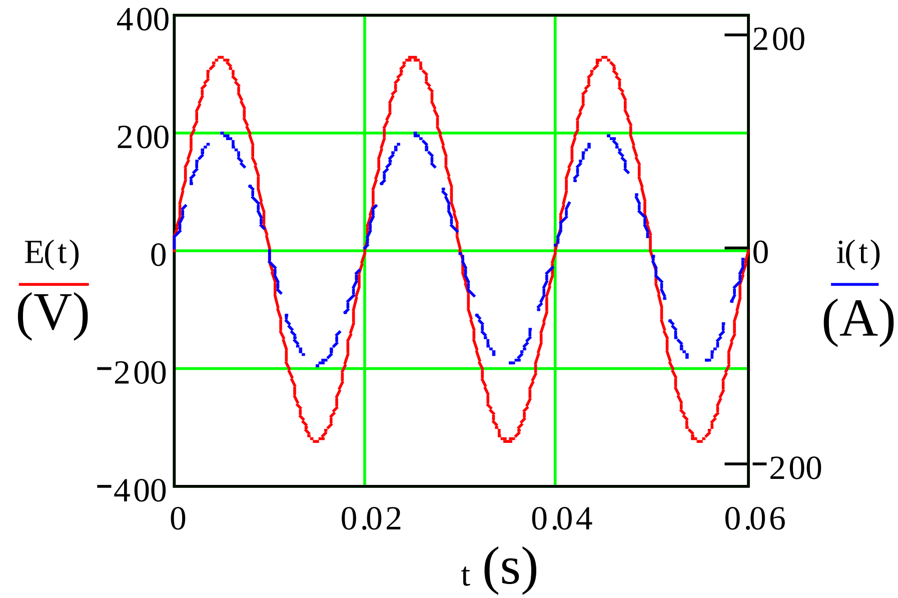
- Now look at some LCR circuits.
- All with $\mathcal{E}_m = 325$ V, f = 50 Hz:
- R = 3Ω, C = 100 mF, L = 10 mH so $X_C = 0.032\ \Omega$ and $X_L = 3.14\ \Omega$.



Series LCR Circuit

- R = 3Ω, C = 1 mF, L = 0.1 mH so X_C = 3.18 Ω and X_L = 0.031 Ω.
- R = 3, C = 1 mF, L = 10 mH so X_C = 3.18 Ω and X_L = 3.14 Ω.





- $X_L > X_C$, more inductive than capacitive, ϕ +ive, i lags v.
- $X_C > X_L$, more capacitive than inductive, ϕ –ive, i leads v.
- $X_L \approx X_C$.
- Notice current large and $\phi \approx 0$.

Series LCR Circuit: Resonance

- Remember:

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}.$$

- Resonance (maximum amplitude of current) when:

$X_L = X_C$ [20.4]

- Recall also:

$$\tan\phi = \frac{X_L - X_C}{R}.$$

- See $\tan\phi = 0$, and hence $\phi = 0$, when $X_L = X_C$.

- Can also change relationship between capacitive and inductive reactances by changing frequency at which circuit is driven.

- Current amplitude given by:

$$I = \frac{\mathcal{E}_m}{Z} = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}.$$

- Maximum when:

$$\omega_d L - \frac{1}{\omega_d C} = 0 \Rightarrow \omega_d = \frac{1}{\sqrt{LC}} \qquad [20.5]$$

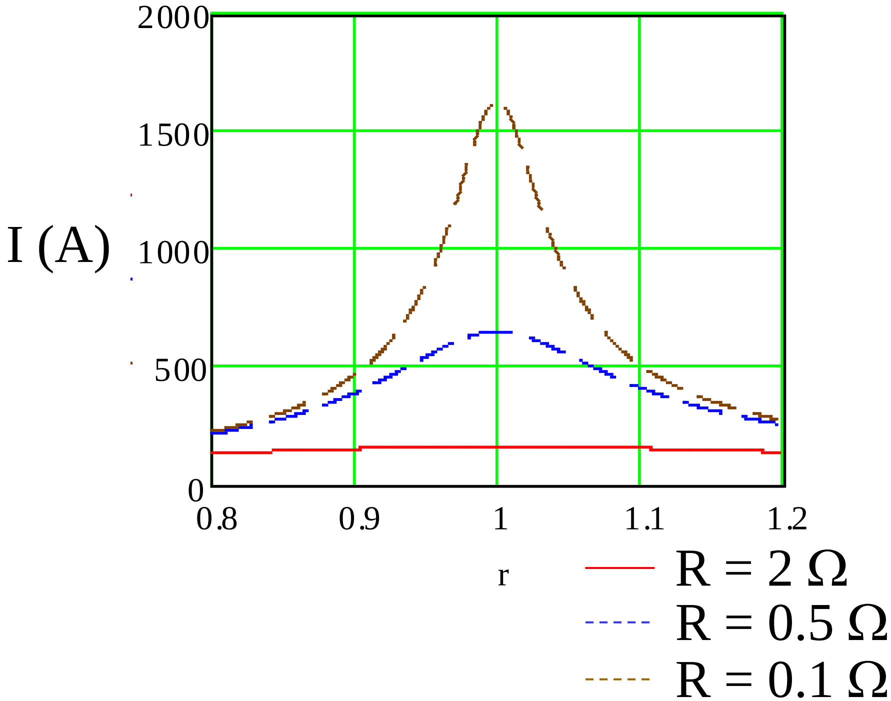
- Recall natural frequency of LCR circuit is:

$$\omega' = \sqrt{\omega^2 - (R/2L)^2} \approx \omega$$

where $\omega = 1/\sqrt{LC}$

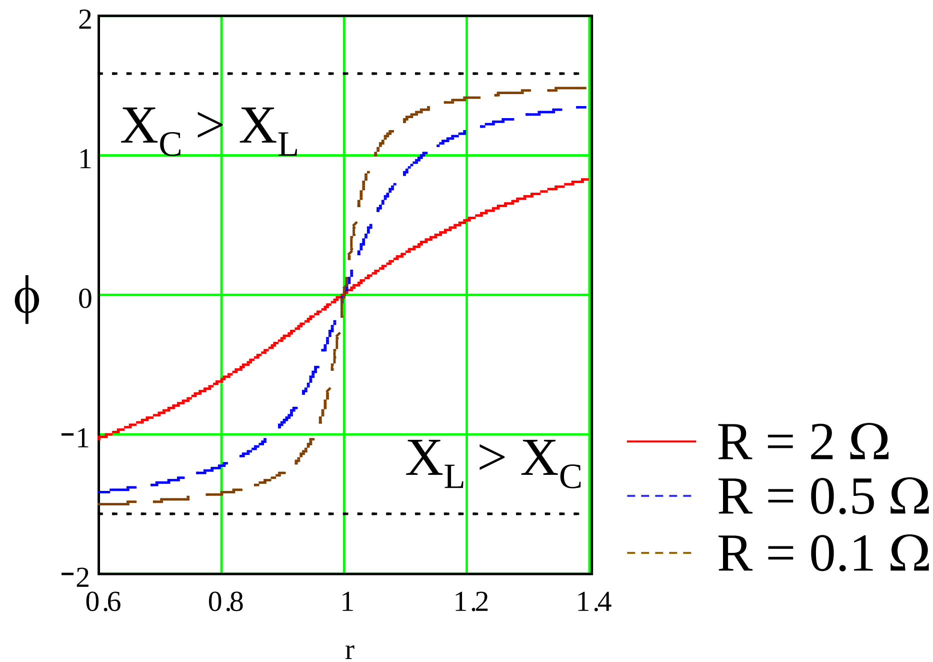
Resonance and Phase

- Plot current amplitude as function of $r = \omega_d/\omega$, for R = 2, 0.5 and 0.2 Ω:



- Maximum always when $\omega_d \approx \omega$, but sharpness of resonance changes.

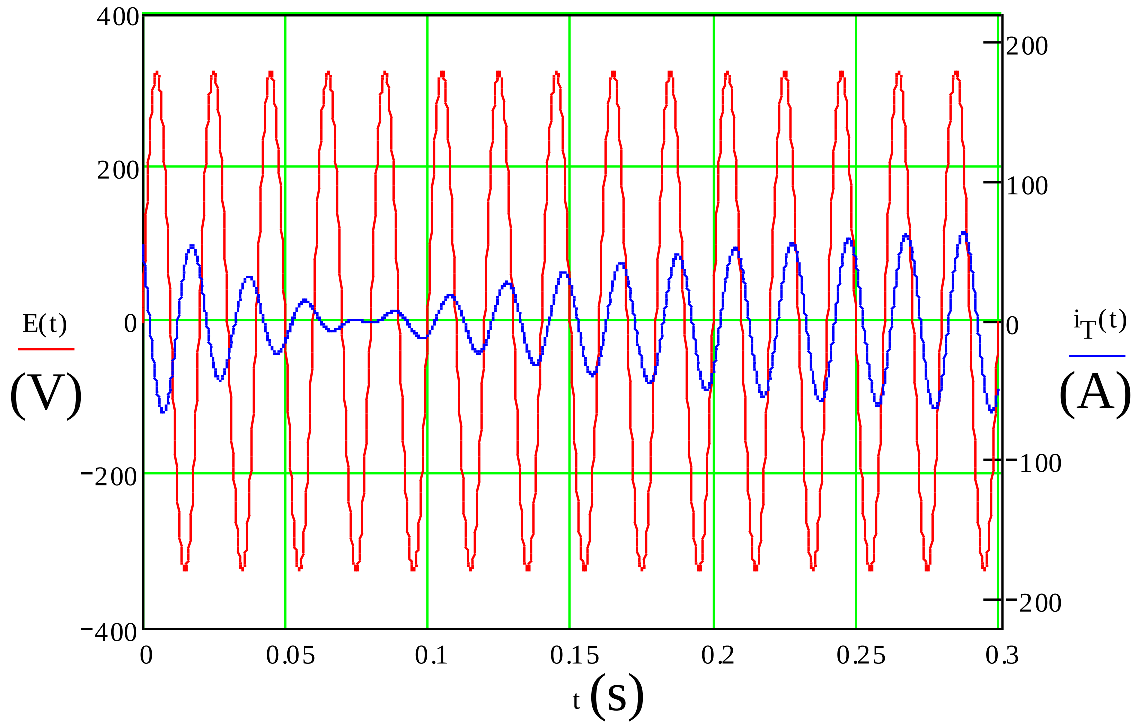
- Phase also change as frequency varies:



- Phase increases from $-\pi/2$, through zero as frequency passes through resonance, to $+\pi/2$.

Transients

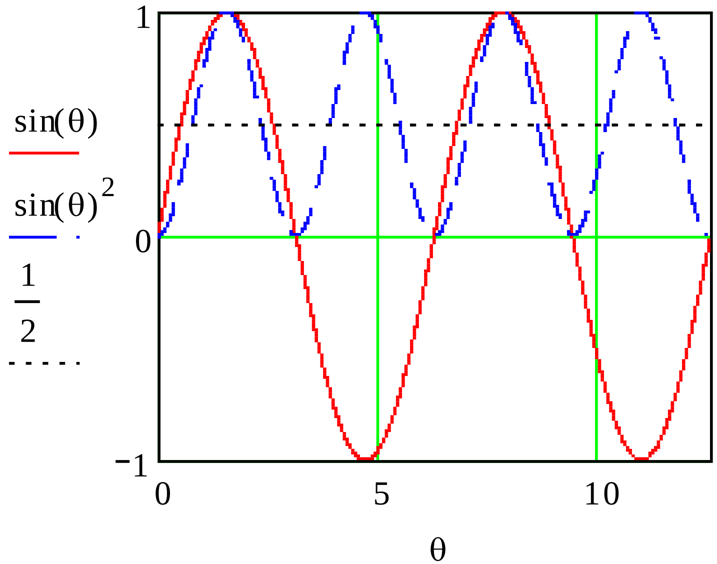
- Note, when the emf is first applied, it can take some time for the current to settle down to the steady state described by the equation $i = I \sin(\omega_d t - \phi)$.
- The additional current that flows initially is called the transient current.
- The speed with which the transient current dies out is determined by the time constants of the circuit $\tau = L/R$ and $\tau = RC$.
- Example of behaviour that might be observed in LCR circuit at "start-up":



Power in AC Circuits

- In LCR circuit energy is shifted between C and L (i.e. E and B fields) and lost to the circuit through the R.
- If have emf, this replaces energy lost through R.
- Power dissipated in R given by:
  $P = i^2 R = I^2 \sin^2(\omega_d t - \phi) R.$
- Mean rate of energy loss is this averaged over time.
- From plot opposite, see average value of $\sin^2\theta$, $\langle \sin^2\theta \rangle = ½$, so:
  $$P_{avg} = I^2 R \left\langle \sin^2(\omega_d t - \phi) \right\rangle$$
  $$= \frac{I^2 R}{2} \qquad [20.6]$$

- Plot of $\sin\theta$ and $\sin^2\theta$:



- Introducing the quantity
  $I_{rms} = I/\sqrt{2} \qquad [20.7]$
  we can write:
  $P_{avg} = I_{rms}{}^2 R \qquad [20.8]$

Power in AC Circuits

- Similarly define $V_{rms} = V/\sqrt{2}$ [20.9]
  and $\mathcal{E}_{rms} = \mathcal{E}_m/\sqrt{2}$ [20.10]
- Remember we have:

$$I = \frac{\mathcal{E}_m}{Z} = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}.$$

- We can divide through by $\sqrt{2}$ to get:

$$\frac{I}{\sqrt{2}} = \frac{\mathcal{E}_m/\sqrt{2}}{Z} = \frac{\mathcal{E}_m/\sqrt{2}}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}.$$

- So we can write:

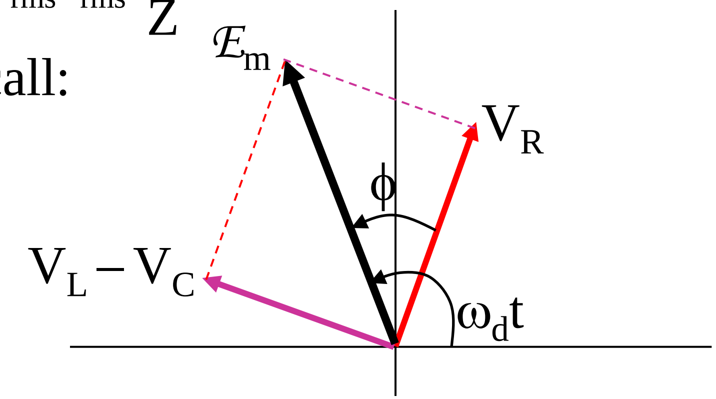
$$I_{rms} = \frac{\mathcal{E}_{rms}}{Z} = \frac{\mathcal{E}_{rms}}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}.$$

- Using this we can recast the expression for the average power:

$$P_{avg} = I_{rms}^2 R = \frac{\mathcal{E}_{rms}}{Z} I_{rms} R$$

$$= \mathcal{E}_{rms} I_{rms} \frac{R}{Z}.$$

- But, recall:



- So $\cos\phi = \frac{V_R}{\mathcal{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}$ [20.11]
- Hence $P_{avg} = \mathcal{E}_{rms} I_{rms} \cos\phi$ [20.12]