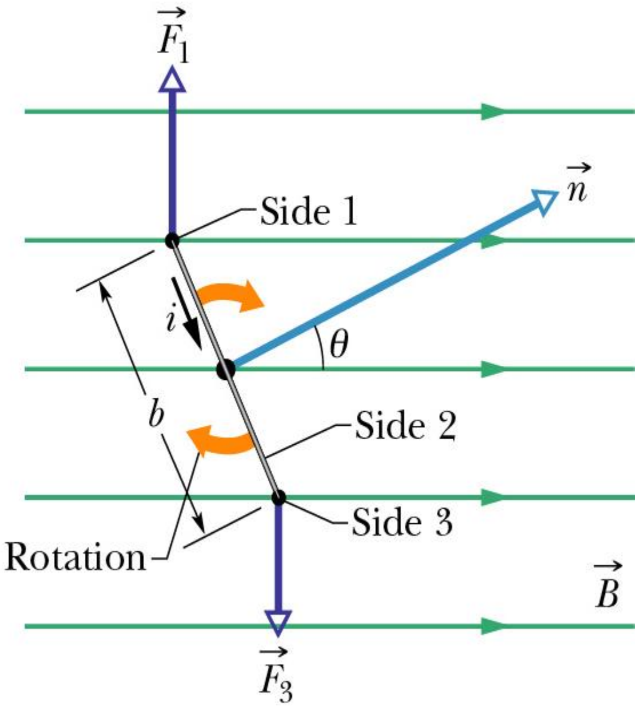
# Lecture 17

- In this lecture we will look at:
  - Electricity generators.
  - Inductors and inductance.
  - Inductance of a solenoid.
  - RL circuits.
  - Energy stored in a B field.
  - Energy density of a B field.
  - Mutual inductance.

- After this lecture, you should be able to answer the following questions:
- What is the emf induced in a coil of area 1 $cm^2$ consisting of 100 turns of wire rotating with a frequency of 60 Hz in a uniform magnetic field of strength 0.5 T.
- How is inductance defined and what is the inductance of a coil with N turns?
- Describe the current that flows in a series circuit containing a resistance, an inductance, a battery and a switch from the point at which the switch is closed.

# Electricity Generators

- Recall electric motor:




- Force on current carrying wire in magnetic field causes rotation of coil.
- Now use external force to rotate the coil in the magnetic field.

- Frequency f, angular frequency $\omega = 2\pi f$.
- Flux through N loops of coil given by:

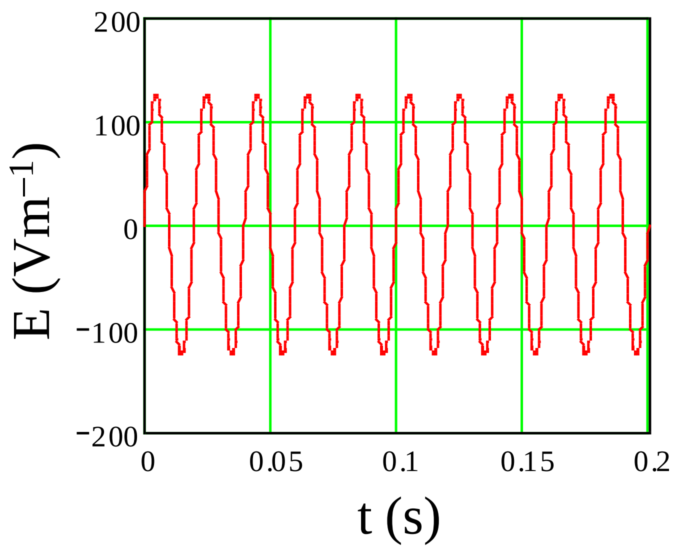
$$\Phi_B = \int \vec{B} \cdot d\vec{A} = BAN\cos\theta \qquad [17.1]$$

- Induced emf from Faraday's law,

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}BAN\cos\omega t$$

$$= BAN\omega\sin\omega t \qquad [17.2]$$

- N=1000, B=1T, A=4x10$^{-4}$m$^2$, f=50Hz.



# Inductors and Inductance: Inductance of Solenoid

- If a current i produces a magnetic flux of $\Phi_B$ in a loop, the inductance is defined to be:

  $$L = \frac{\Phi_B}{i} \qquad [17.3]$$

- Unit is the henry ($H = T\ m^2\ A^{-1}$).
- If the inductor is a coil made of N loops:

  $$L = \frac{N\Phi_B}{i} \qquad [17.4]$$

- (All the windings are linked by the shared flux $\Phi_B$, the product $N\Phi_B$ is called the magnetic flux linkage.)
- Assume no magnetic materials in vicinity of inductors considered in following.
- Consider solenoid with n turns per unit length, area A.
- Magnetic field, $B = \mu_0 in$.
- For length x far from ends, $N\Phi_B = (nx)(BA)$.
- Hence:

  $$L = \frac{N\Phi_B}{i} = \frac{(nx)(BA)}{i}$$

  $$= \frac{(nx)(\mu_0 in)(A)}{i} = \mu_0 n^2 xA.$$

- The inductance per unit length (far from ends of solenoid) is:

  $$\frac{L}{x} = \mu_0 n^2 A.$$

- For solenoid with total length X >> r:

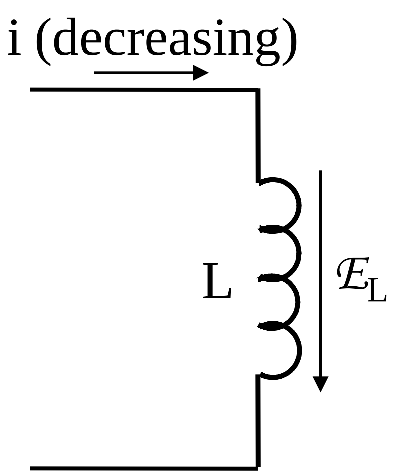
  $$L = \mu_0 n^2 AX \qquad [17.5]$$

# Inductors and Inductance

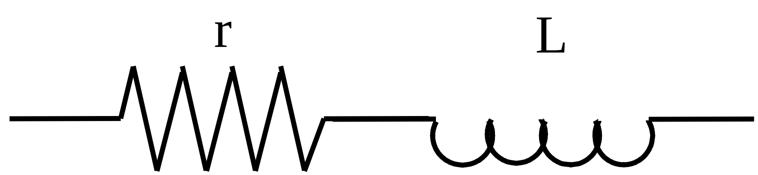
- Current change (e.g. through solenoid) can cause change of magnetic flux and hence induce emf.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} iL = -L\frac{di}{dt} \qquad [17.6]$$

- This is self-inductance, current i changes and self-induced emf appears across solenoid.

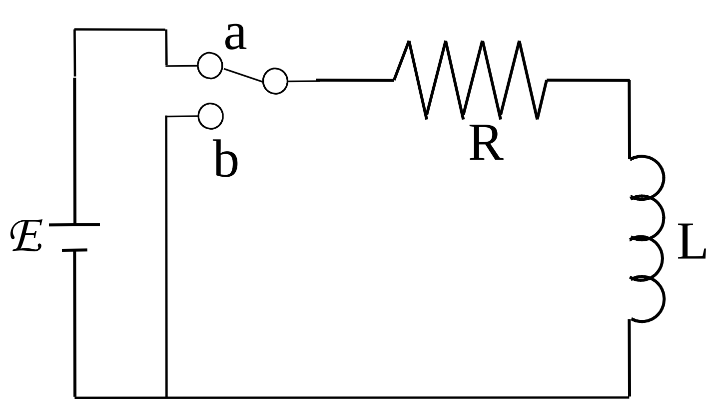



- Direction of induced emf given by Lenz's law.
- In example, emf must oppose decrease in current, so tends to produce current in indicated direction.
- Potential across inductor, $V_L$, is equal to $\mathcal{E}_L$ if resistance of inductor is negligible.
- If inductance has internal resistance r, split into "perfect" inductor and resistor:



# RL Circuits

- Consider following circuit:




- Use Kirchoff's loop rule:

$$-iR - L\frac{di}{dt} + \mathcal{E} = 0 \text{ or}$$

$$L\frac{di}{dt} + iR = \mathcal{E} \qquad [17.7]$$

- (Compare with equation for RC circuit!)

- Solution:

$$i = \frac{\mathcal{E}}{R}\left(1 - \exp\left(-\frac{Rt}{L}\right)\right) \qquad [17.8]$$

- Inductive time constant

$\tau = L/R \qquad [17.9]$

- Move switch to position b, Kirchoff's loop rule then gives:

$$-iR - L\frac{di}{dt} = 0 \text{ or}$$

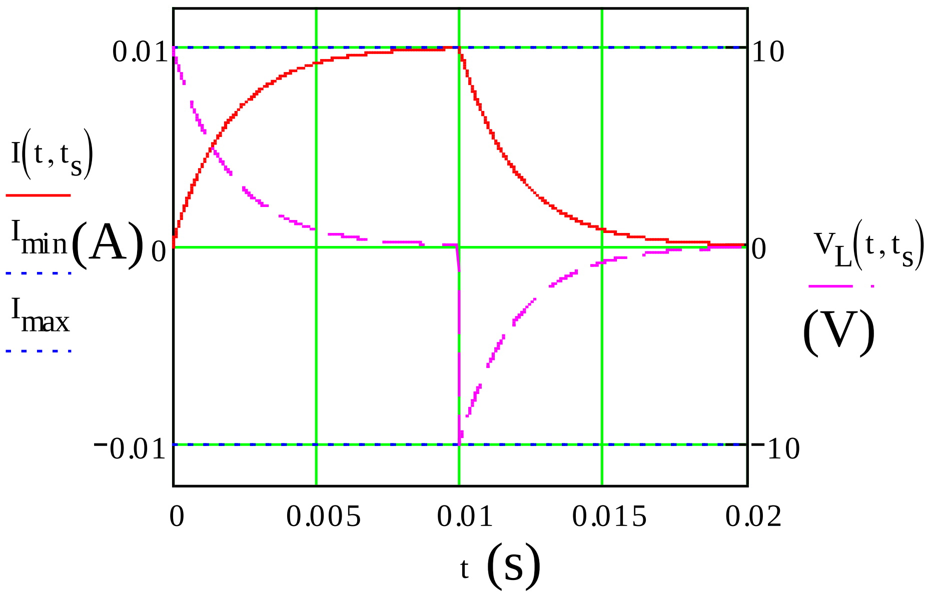
$$L\frac{di}{dt} + iR = 0 \qquad [17.10]$$

- Solution now:

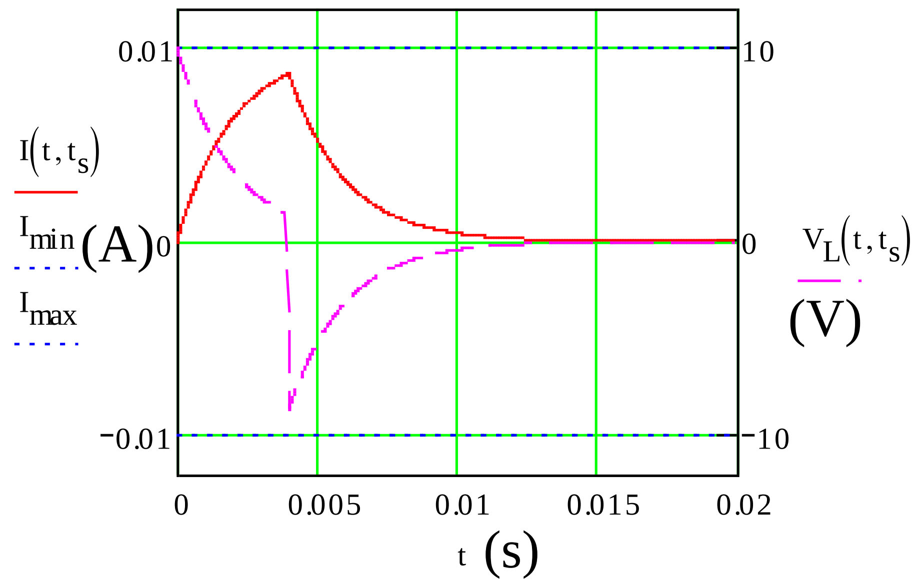
$$i = \frac{\mathcal{E}}{R}\exp\left(-\frac{Rt}{L}\right) \qquad [17.11]$$

# RL Circuits

- Example, R = 1000 Ω, L = 2 H, $\mathcal{E}$ = 10 V, $\tau$ = L/R = 0.002 s.
- Switch initially in position a then move to position b after 5$\tau$ = 0.01 s.
- Look at current through and potential across inductor.

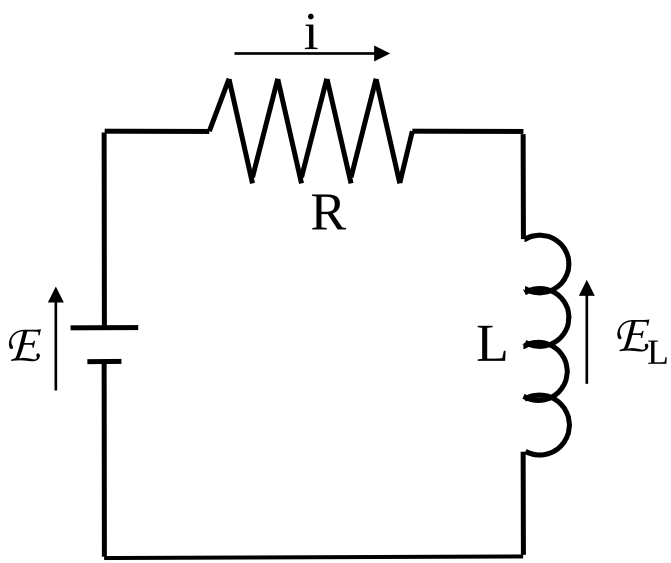



- If move switch after only 2$\tau$ = 0.004 s:



# Energy Stored in a Magnetic Field

- Energy can be stored in a B field.
- Consider again RL circuit:




- Differential equation describing circuit:

$$\mathcal{E} = L\frac{di}{dt} + iR.$$

- Multiply each side by i:

$$\mathcal{E}i = Li\frac{di}{dt} + i^2R.$$

- Now if charge dq passes through battery in time dt, rate at which work is done on it is $(\mathcal{E}dq)/dt = \mathcal{E}i$, so LHS is rate at which battery delivers energy to rest of circuit.
- Term $i^2R$ is rate at which energy dissipated in resistor.
- Remainder must be rate at which energy is stored in magnetic field, i.e.

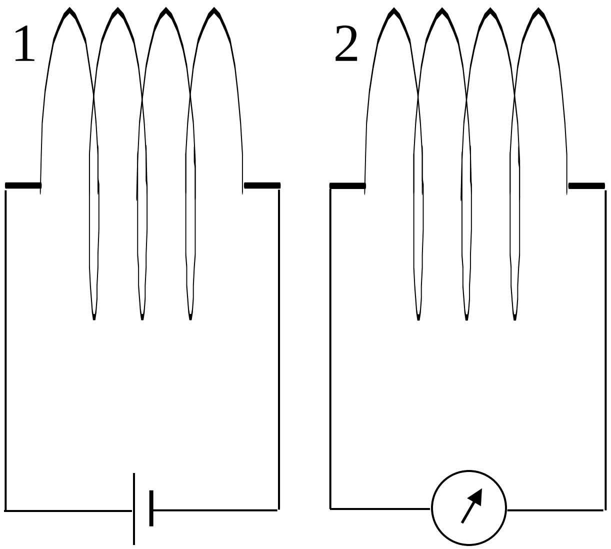
$$\frac{dU_B}{dt} = Li\frac{di}{dt}.$$

# Energy Stored in, and Energy Density of, B Field

- Rewriting
  $$\frac{dU_B}{dt} = Li\frac{di}{dt},$$
  we get $dU_B = Li\,di.$
- This can then be integrated:
  $$\int dU_B = \int Li\,di$$
  $$\Rightarrow U_B = \tfrac{1}{2}Li^2 + \text{const.}$$
- Setting the energy to be zero for zero magnetic field (i.e. zero current):
  $$U_B = \tfrac{1}{2}Li^2 \qquad [17.12]$$
- Compare with energy stored in charged capacitor:
  $$U_E = \tfrac{1}{2}CV^2.$$
- Consider length x far from ends of solenoid of volume Ax with n turns per unit length.
- The energy stored in this volume is:
  $$u_B = \frac{U_B}{Ax} = \frac{\tfrac{1}{2}Li^2}{Ax} = \frac{L}{x}\frac{i^2}{2A}.$$
- Now
  $$\frac{L}{x} = \mu_0 n^2 A \text{ so } u_B = \tfrac{1}{2}\mu_0 n^2 i^2.$$
- We also know $B = \mu_0 i\,n$, so energy density:
  $$u_B = \frac{B^2}{2\mu_0} \qquad [17.13]$$
- Compare with energy density of electric field:
  $$u_E = \tfrac{1}{2}\varepsilon_0 E^2.$$

# Mutual Induction

- Consider two coils close together.




- Current in coil 1 induces magnetic field which passes through (links) coil 2.
- If change current through 1, $i_1$, change field and hence flux through coil 2, inducing emf in coil 2.

- Flux through coil 2 is $\Phi_{21}$ (i.e. flux through 2 generated by current in 1).
- Define mutual inductance:

$$M_{21} = \frac{\Phi_{21}}{i_1} \qquad [17.14]$$

- Rewriting the above: $M_{21} i_1 = \Phi_{21}$.
- Now vary $i_1$ with time:

$$M_{21}\frac{di_1}{dt} = \frac{d\Phi_{21}}{dt}$$

- Faraday's law tells us RHS is magnitude of emf appearing across coil 2, so we write (with – sign due to Lenz's law):

$$\mathcal{E}_2 = -M_{21}\frac{di_1}{dt}.$$

# Mutual Induction

- Can make same argument for emf induced in coil 1 due to current in coil 2, find:

  $$\mathcal{E}_1 = -M_{12}\frac{di_2}{dt}.$$

- The emf induced in either coil is proportional to the change in current in the other coil.
- It turns out that $M_{12} = M_{21} \equiv M$.
- Hence we can write:

  $$\mathcal{E}_1 = -M\frac{di_2}{dt} \text{ and}$$

  $$\mathcal{E}_2 = -M\frac{di_1}{dt} \qquad [17.15]$$

- The unit of mutual inductance is the same as that of self inductance, the henry.