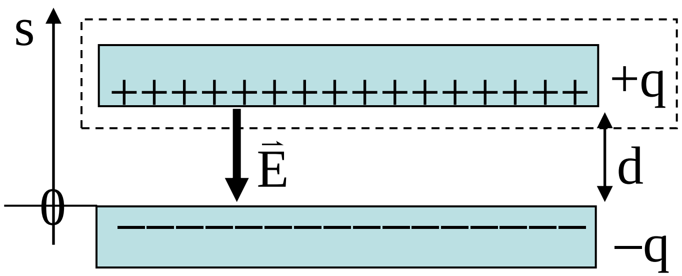
Lecture 7

- This lecture, we will look at:
  - Capacitance.
  - Calculating capacitances for simple shapes/configurations of electrodes.
  - Adding capacitances in series and parallel.

- After this lecture, you should be able to answer the following questions:
- What is the capacitance of a sandwich consisting of 2 sheets of aluminium of dimensions 1×1 m^2 separated by 100 μm?
- How large would the aluminium plates have to be for the capacitance of this device to be equal to that of the earth?

Relating Charge and Potential: Capacitance

- Parallel plate capacitor.
- Plates area A, separation d.



- Magnitude of E field due to charge +q on upper plate from Gauss' law:

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$EA = \frac{q}{\varepsilon_0} \text{ or } E = \frac{q}{\varepsilon_0 A} \qquad [7.1]$$

- Potential difference between plates:

$$V = -\int_0^d \vec{E}\cdot d\vec{s} = E\int_0^d ds = Ed.$$

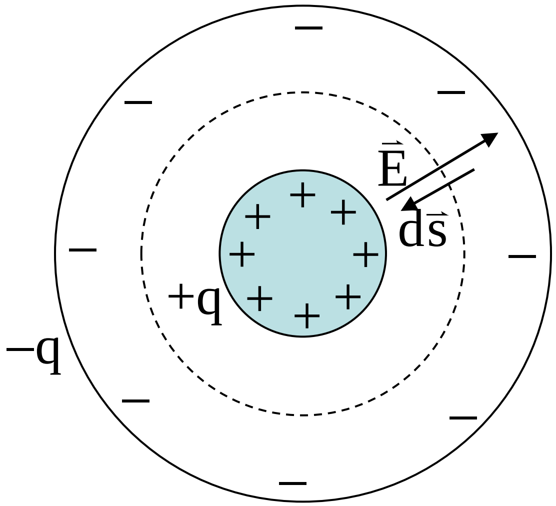
- Using the previous expression for E:

$$V = q\frac{d}{A\varepsilon_0}.$$

- Potential difference proportional to charge.
- We define the capacitance C so
  V = q/C [7.2]
- Hence, $C = q/V$ and, for the parallel plate capacitor, $C = \frac{A\varepsilon_0}{d}$.
- Units Farads (F).
- See also see that E = V/d [7.3]
  (applies where E field is uniform).

Calculating Capacitance

- Capacitance of coaxial cylinders.
- End view of cylinders, length L, inner radius a, outer radius b.



- Gauss' law (cylindrical surface):

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow EA = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{\varepsilon_0 A} = \frac{q}{\varepsilon_0 (2\pi r L)} = \frac{q}{2\pi\varepsilon_0 L r}.$$

- Now we can work out the potential difference between the cylinders:

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{s} = \frac{-q}{2\pi\varepsilon_0 L}\int_b^a \frac{dr}{r}$$

$$= \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right).$$

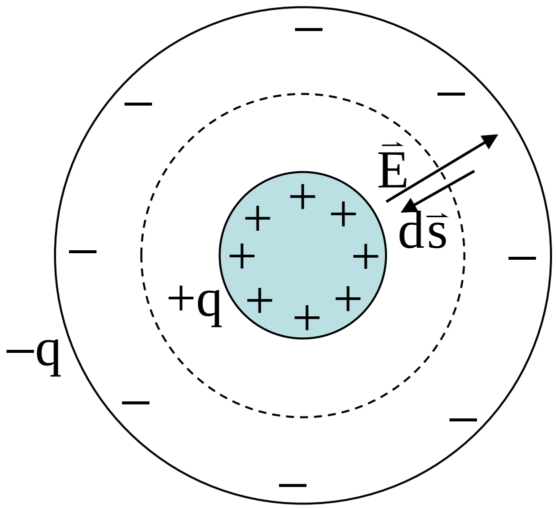
- Using C = q/V we have:

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)}.$$

- Important result, applies to coaxial cables!

Calculating Capacitance

- Spherical Capacitor.
- Two concentric spherical shells, inner radius a, outer radius b.



- Gauss' law (spherical surface):

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow EA = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{\varepsilon_0 A} = \frac{q}{\varepsilon_0 (4\pi r^2)} = \frac{q}{4\pi\varepsilon_0 r^2}.$$

- Now we can work out the potential difference between the cylinders:

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{s} = \frac{-q}{4\pi\varepsilon_0}\int_{b}^{a}\frac{dr}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{q}{4\pi\varepsilon_0}\frac{b-a}{ab}.$$

- Using C=q/V we have:

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}.$$

- Note always have expression of form capacitance ~ ε_0 x (something with dimensions of length).
- Hence units of ε_0 Farads per metre.

Calculating Capacitance

- Isolated sphere:

- Rewrite result for concentric spheres:

$$C = 4\pi\varepsilon_0 \frac{a}{1 - a/b}.$$

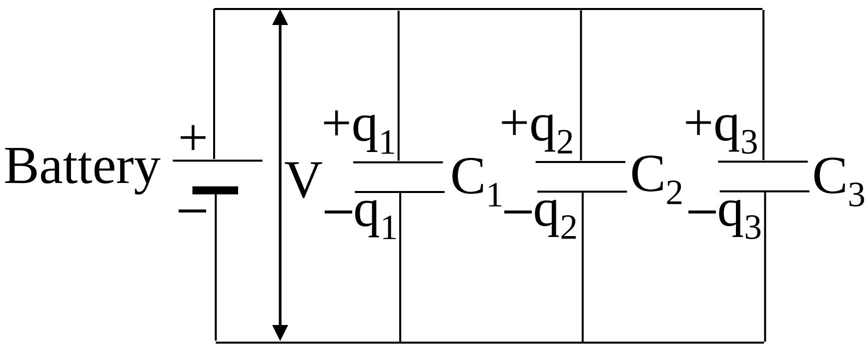
- Let $b \to \infty$: $C = 4\pi\varepsilon_0 a.$
- Capacitance of the earth:

$$C = 4\pi \times 8.85 \times 10^{-12} \times 6.37 \times 10^{6}$$
$$= 7.08 \times 10^{-4}\ \text{F}.$$

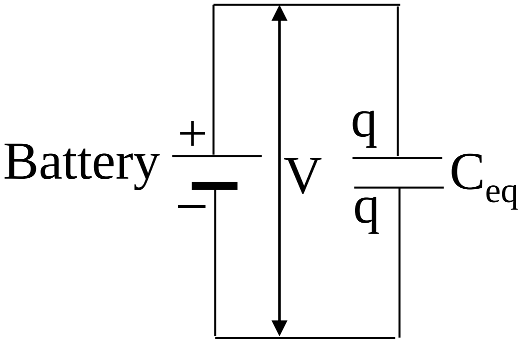
- Can now buy “ultra-capacitors” with $C \sim 150\ \text{F}$!

Capacitances in Parallel

- Capacitors used in electrical circuits.
- What is combined effect of capacitances in parallel?



- Look for C_{eq} which replaces capacitors above:



- Want $q = q_1 + q_2 + q_3$.
- Now $q_1 = C_1V$, $q_2 = C_2V$ and $q_3 = C_3V$.
- Hence $q = C_1V + C_2V + C_3V$.
- This gives:

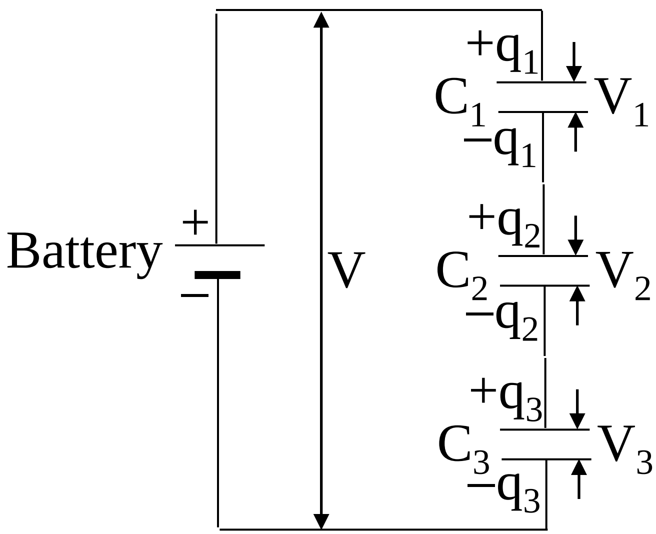
$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3.$$

- So capacitances in parallel add according to:

$$C_{eq} = C_1 + C_2 + C_3 + \dots \qquad [7.4]$$

Capacitances in Series

- Find equivalent capacitance for series circuit:



- Now $q_1 = q_2 = q_3 = q$.
  (The battery "pushes" electrons onto the bottom plate of C_3, which repel the electrons in the top plate of C_3 onto the bottom plate of C_2...)

- The potential differences across each of the capacitors are:

$$V_1 = \frac{q}{C_1},\ V_2 = \frac{q}{C_2} \text{ and } V_3 = \frac{q}{C_3}.$$

- But $V = V_1 + V_2 + V_3$ so we have:

$$V = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_2}.$$

- Hence:

$$C_{eq} = \frac{q}{V} = \frac{q}{q/C_1 + q/C_2 + q/C_3}$$

- Rewriting we see:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \qquad [7.5]$$