Solutions September 2020 PHYS208

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Q1a

Give brief explanations that distinguish the differences between each of the following pairs of statistical terms:

(i) *systematic* and *random* errors of an experiment

Ans. Systematic errors - non-repeatable, non-statistical fluctuation or offset on values, that does not obey random statistics.
Random errors - fluctuations usually drawn from a Gaussian distribution; mean converges to true value with repeated measurements.

(ii) *standard deviation* and *standard error* estimations

Ans - standard deviation is an estimate of the spread or width of a distribution about the mean;
standard error is an estimate of the uncertainty in the knowledge of the mean from the combination of all measurements (could give mathematical definitions but not required).

(iii) *subjective* and *empirical* probabilities.

Ans - subjective probability is a degree of belief usually based on prior knowledge, and is inherent in a Bayesian analysis of probability; empirical probability is based on repeated experimental outcomes alone, and is part of the frequentist analysis of probability. B+U3

A probability density $P(x)$ is given by

$$P(x) = 6x - 6x^2$$

over the region $0 \le x \le 1$. Find the expectation value $E(x)$ and the variance $V(x)$.

Ans - for $E(x)$, *integrate* $xP(x)$ *giving* $2x^3 - \frac{3}{2}x^4$. *Put in limits* $x = 0, 1$ *to give a value* $E(x) = 2 - 3/2 = 0.5$
For $V(x)$, *first evaluate the integral of* $x^2P(x)$ *over the same range, giving* $[\frac{6}{4}x^4 - \frac{6}{5}x^5]_0^1 = 3/2 - 6/5 = 3/10$. U2
Variance $= E(x^2) - E(x)^2 = 3/10 - 1/4 = 0.05$.

Q1b

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1b

$$U = e^{i\rho S} = \begin{pmatrix} \cosh\rho & -\sinh\rho \\ -\sinh\rho & \cosh\rho \end{pmatrix}$$ B2

B2

$$\left.\frac{dU}{d\rho}\right|_{\rho=0} = iS \rightarrow S = -i\left.\frac{dU}{d\rho}\right|_{\rho=0}$$ B1

B1

$$\cosh\rho = \frac{1}{2}\left[e^{\rho}+e^{-\rho}\right], \quad \sinh\rho = \frac{1}{2}\left[e^{\rho}-e^{-\rho}\right]$$

$$\frac{d}{d\rho}\cosh\rho = \sinh\rho \qquad \frac{d}{d\rho}\sinh\rho = \cosh\rho$$ U1

U1

$$S = -i\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_1$$ B1+U1

B1+U1

$$\cosh\rho = \gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow U = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$ B1+U1

B1+U1

$$\sinh\rho = \beta\gamma$$

$$\beta = \tanh\rho = \frac{1-e^{-2\rho}}{1+e^{-2\rho}} \leq 1$$ U2

U2

Q1c

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1c

$$\|f\|^2 = \int_{-1}^{1} [i + x + 2x^2][-i + x + 2x^2]\, dx$$ B1

$$= \int_{-1}^{1} [1 + x^2 + \cancel{4x^3} + 4x^4]\, dx$$

$$= x + \frac{1}{3}x^3 + \frac{4}{5}x^5 \Big|_{-1}^{1}$$ U1

$$= 2 + \frac{2}{3} + \frac{8}{5} = \frac{64}{15}$$ U1

$$\|f\| = \sqrt{\frac{64}{15}} = 2.066$$

use $g = i + x$
$h = e^{ix}$

$$\int_{-1}^{1} (1+ix)\, e^{-ix}\, dx = g \cdot h = \int_{-1}^{1} g \cdot h^* \, dx$$ B1

$$\frac{d}{dx}\left[x e^{\alpha x}\right] = e^{\alpha x} + \alpha x e^{\alpha x} \xrightarrow[\alpha = -i]{} x \cdot e^{-ix} = -\frac{1}{i}\frac{d}{dx}\left[x e^{-ix}\right] + \frac{1}{i} e^{-ix}$$ U1

$$g \cdot h = -\frac{1}{i} e^{-ix}\Big|_{-1}^{1} - \frac{i}{i}\left(x e^{-ix}\right)\Big|_{-1}^{1} + \frac{i}{i}\left(\frac{-1}{i}\right) e^{-ix}\Big|_{-1}^{1}$$

$$= i\left(e^{-i} - e^{i}\right) - 1\left(e^{-i} + e^{i}\right) + \left(e^{-i} - e^{i}\right) \cdot i$$

$$= (2i - 1)e^{-i} + (-2i - 1)e^{i}$$ U2

$$= \left(e^{i} + e^{-i}\right) \cdot 1 + \left(e^{-i} - e^{i}\right) \cdot 2i$$

Q1d

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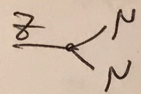
1d)

$$p^2 = M^2 = E^2 - \vec{p}^{\,2} \rightsquigarrow |\vec{p}| = \sqrt{E^2 - M^2}$$ B1



$$: \quad M_Z^2 = (p_{\nu_1} + p_{\nu_2})^2$$

$$= p_{\nu_1}^2 + p_{\nu_2}^2 + 2E_1E_2 - 2\sqrt{E_1^2 - M_\nu^2}\sqrt{E_2^2 - M_\nu^2}\cos\vartheta$$

$$M_Z^2 = 2M_\nu^2 + 2E_1E_2 - 2\sqrt{E_1^2 - M_\nu^2}\sqrt{E_2^2 - M_\nu^2}\cos\vartheta.$$ U2

$$\approx 2E_1E_2(1 - \cos\vartheta) = 2E^2(1 - \cos\vartheta).$$

$$\cos\vartheta = 1 - \frac{M_Z^2}{2E^2}$$ U1

$M_Z = 91.1$ GeV

$E = 150$ GeV

$\cos\vartheta = 0.816$

$\vartheta = 35{,}4^\circ$ U2

Q1e

all unseen

1e

$$\frac{dy}{dx} = z + 2x \qquad \frac{dz}{dx} = y + x$$

$$z = \frac{dy}{dx} - 2x \qquad \frac{dz}{dx} = \frac{d^2y}{dx^2} - 2 = y + x$$

[3]

$$\frac{d^2y}{dx^2} - y = 2 + x$$

homogeneous eq: $\frac{d^2y}{dx^2} - y = 0$

$y = Ae^{Bx}$: $AB^2e^{Bx} - Ae^{Bx} = 0 \Rightarrow B^2 = 1, \; B = \pm 1$

[3]

$y_h = Ae^{x} + Ce^{-x}$

particular solution: try $y = Dx^2 + Ex + F$

$2D - 2 = Dx^2 + Ex + F + x$

$\rightsquigarrow D = 0, \; E = -1, \; F = -2.$

[3]

$$y = Ae^{x} + Ce^{-x} - x - 2$$

$y(0) = A + C - 2 = 2 \rightarrow A + C = 4$: $A = 3$

$y'(0) = A - C - 1 = 1 \qquad A - C = 2$: $C = 1$

[1]

$y = 3e^{x} + e^{-x} - x - 2$, $z = 3e^{x} - e^{-x} - 1 - 2x$ [2]

Q2Aa

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all unseen

2Aa $y'' + 2y' + y = 4e^{-x}$

[2]

homogeneous : $y''+2y'+y=0$

$y_{1,h} = ae^{bx}$: $ab^2 + 2ab + a = 0 \Rightarrow b^2+2b+1=0$: $b=-1$

$= ae^{-x}$ [3]

$y_{2,h} = Ax^{B}e^{Cx}$:

$x^{B}\left[A + 2AC + AC^2\right] + x^{B-1}\left[2AB + 2ABC\right] + x^{B-2}\left[AB(B-1)\right] = 0$

$A(1+2C+C^2)$ ↳ $C=-1$; $B=1$

$= Axe^{-x}$ $\rightarrow y_h = (a+bx)e^{-x}$ [3]

inhomogeneous

$y = \alpha x^{\beta} e^{-x}$ [2]

$x^{\beta}[\cancel{\alpha - 2\alpha + \alpha}] + x^{\beta-1}[\cancel{2\alpha\beta - 2\alpha\beta}] + x^{\beta-2}[\alpha\beta(\beta-1)] = 4$

$\underline{\beta = 2}$, $2\alpha = 4 \rightarrow \underline{\alpha = 2}$

Complete : $y = (a+bx)e^{-x} + 2x^2e^{-x}$

$y(0)=1$: $a=1$

$y(1)=\frac{1}{e}$: $(1+b+2)=1 \rightarrow b=-2$

$\underline{y = (1-2x+2x^2)e^{-x}}$, $\underline{y(2) = 5e^{-2}}$ [2]

Q2Ab

General starting point, including breaking up integral into two pieces $[-1,0]$ and $[0,1]$

$$\begin{aligned}\lambda_i = P_i(x)\cdot f(x) &= \int_{-1}^{1}(P_i(x))^* f(x)\mathrm{d}x \\ &= \int_{-1}^{0}(P_i(x))^*(-1)\mathrm{d}x + \int_{0}^{1}(P_i(x))^*(+2)\mathrm{d}x\end{aligned}$$ [B1]

Calculation of λ_0

$$\begin{aligned}\lambda_0 &= \int_{-1}^{0} -\frac{1}{\sqrt{2}}\mathrm{d}x + \int_{0}^{1}\frac{2}{\sqrt{2}}\mathrm{d}x \\ &= -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707\end{aligned}$$ [U1]

Calculation of λ_1

$$\begin{aligned}\lambda_1 &= \int_{-1}^{0} -\frac{1}{\sqrt{2}}e^{-i\pi x}\mathrm{d}x + \int_{0}^{1} 2\frac{1}{\sqrt{2}}e^{-i\pi x}\mathrm{d}x \\ &= -\left[\frac{e^{-i\pi x}}{-\sqrt{2}i\pi}\right]_{-1}^{0} + \left[2\frac{e^{-i\pi x}}{-\sqrt{2}i\pi}\right]_{0}^{1} \\ &= -\left(\frac{ie^{0}}{\sqrt{2}\pi} - \frac{ie^{i\pi}}{\sqrt{2}\pi}\right) + \left(\frac{2ie^{-i\pi}}{\sqrt{2}\pi} - \frac{2ie^{0}}{\sqrt{2}\pi}\right) \\ &= -\frac{i}{\sqrt{2}\pi} - \frac{i}{\sqrt{2}\pi} - \frac{2i}{\sqrt{2}\pi} - \frac{2i}{\sqrt{2}\pi} = \frac{-3\sqrt{2}i}{\pi}\end{aligned}$$ [U2]

Calculation of λ_{-1}

$$\begin{aligned}\lambda_{-1} &= \int_{-1}^{0} -\frac{1}{\sqrt{2}}e^{i\pi x}\mathrm{d}x + \int_{0}^{1} 2\frac{1}{\sqrt{2}}e^{i\pi x}\mathrm{d}x \\ &= -\left[\frac{e^{i\pi x}}{\sqrt{2}i\pi}\right]_{-1}^{0} + \left[2\frac{e^{i\pi x}}{\sqrt{2}i\pi}\right]_{0}^{1} \\ &= -\left(\frac{-ie^{0}}{\sqrt{2}\pi} - \frac{-ie^{-\sqrt{2}i\pi}}{\sqrt{2}\pi}\right) + \left(\frac{-2ie^{i\pi}}{\sqrt{2}\pi} - \frac{-2ie^{0}}{\sqrt{2}\pi}\right) \\ &= \frac{i}{\sqrt{2}\pi} + \frac{i}{\sqrt{2}\pi} + \frac{2i}{\sqrt{2}\pi} + \frac{2i}{\sqrt{2}\pi} = \frac{3\sqrt{2}i}{\pi}\end{aligned}$$ [U2]

Thus the approximation of f reads as:

$$\begin{aligned}\lambda_0 P_0(x) + \lambda_1 P_1(x) + \lambda_{-1}P_{-1}(x) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{3\sqrt{2}i}{\pi}\frac{1}{\sqrt{2}}e^{i\pi x} + \frac{3\sqrt{2}i}{\pi}\frac{1}{\sqrt{2}}e^{-i\pi x} \\ &= \frac{1}{2} - \frac{3i}{\pi}\left(e^{i\pi x} - e^{-i\pi x}\right) \\ &= \frac{1}{2} + \frac{6}{\pi}\sin(\pi x)\end{aligned}$$ [U2]

Q2Ba

Briefly describe the use of maximum likelihood as a method of parameter fitting. Under what assumption does maximum likelihood reduce to a χ^2 analysis of data? **[3]**

Ans - Maximum likelihood tests how likely data values are, given a specific model. Maximum likelihood gives the most probable values of model parameters for a given data set. B1 *The likelihood function is the product of the probabilities of all data points generated from a given model.* B1
Maximum likelihood is equivalent to a χ^2 analysis when errors are Gaussian. B1

Q2Bb

Ten temperatures are measured, each with an error of 0.2 K:

No.	T(K)
1	10.3
2	10.2
3	9.7
4	10.5
5	9.6
6	9.9
7	10.2
8	10.0
9	10.1
10	9.9

It is suggested that these measurements are all consistent with the mean value, *estimated from the data*, of 10.04, and that the differences are simply due to the measurement errors. Find χ^2 and the number of degrees of freedom.

Ans - sum $(value - 10.04)^2/(0.2)^2$ *over the ten points* $= 1.69 + 0.64 + 2.89 + 5.29 + 4.84 + 0.49 + 0.64 + 0.04 + 0.09 + 0.49 = 17.7$ U2

U1

Number of degrees of freedom $= N - p = 10 - 1 = 9$.

What do you conclude about the suggestion (use the attached table of critical χ^2 values)?

Ans - from table of critical χ^2 values, suggestion is excluded at the 5% level and can be considered unlikely. U1

How would your conclusion change, if at all, if the original suggestion were that all the values were consistent with a *pre-determined* mean value of 10.04?

(Show all the working of your calculations.) **[5]**

Ans - using a pre-determined mean, $p = 0$, and we have $10 - 0 = 10$ degrees of freedom. From the table, this is only excluded at the $5 - 10\%$ level; marginally inconsistent but not firmly excluded. U1

Q2Bc

Give an example from physics of a process that obeys Poisson statistics.

Ans - any counting experiment, e.g. radioactive decay, photon counting etc. U2

Q2Bd

The Poisson probability distribution is given by:

$$P(r)dL = \frac{\mu^r e^{-\mu}}{r!}$$

Explain the meaning of the terms μ and r.

Ans - μ is the average rate of the events B1
r is the number of successes or events.

The only taxi company operating in an isolated town has twelve taxis. The average number of people in the town who want to travel by taxi at any one time is 10, and you can assume that this usage obeys a Poisson frequency distribution.

(i) On what proportion of occasions is at least one of the taxis in use?

Ans- $p(r \geq 1) = 1 - p(0)$; $\mu = 10$
$p(0) = \frac{10^0 e^{-10}}{0!} = 4.54 \times 10^{-5}$ U2
so one is free $1 - 4.54 \times 10^{-5} = 0.99995$ *of the time.*

(ii) On what proportion of occasions will a traveller find a taxi free for them to make an immediate journey?

Ans - all 12 being in use implies $r = 12, \mu = 10$
$p(12) = \frac{10^{12} e^{-10}}{12!} = 0.0948$ U1
$p(\geq 1 free) = 1 - p(12) = 90.52\%$

(iii) How many taxis would the company need to have to increase the probability in (ii) to more than 95%?

Ans - by working through increasing values of r, find $p(15) = 0.0347$, i.e. probability of a taxi being free is 96.5% if there are 15 in total; it is below 95% for any number smaller than this. U1

Q2Be

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2Be

B1

$$D^2 = \int_{-1}^{1} (ax - e^{3x})\cdot(ax - e^{3x})\,dx$$

$$= \int_{-1}^{1} \left[a^2x^2 - 2axe^{3x} + e^{6x}\right] dx$$

$$= \frac{a^2}{3}\cdot 2 + \frac{1}{6}\left[e^{6} - e^{-6}\right] - 2a\int_{-1}^{1} x e^{3x}\,dx$$

$$\frac{1}{3}\frac{d}{dx}\left[xe^{3x}\right] - \frac{1}{3}e^{3x} = xe^{3x}$$

$$-\frac{2}{3}a\left[xe^{3x}\right]_{-1}^{1} + \frac{2a}{3}\,\frac{1}{3}e^{3x}\Big|_{-1}^{1}$$

$$D^2 = \frac{2}{3}a^2 + \frac{1}{6}\left[e^{6} - e^{-6}\right] - \frac{2}{3}a\left[e^{3} + e^{-3}\right] + \frac{2a}{3}\,\frac{1}{3}\left[e^{3} - e^{-3}\right]$$

$$= a^2\cdot\frac{2}{3} + a\frac{2}{3}\left[\frac{1}{3}(e^{3} - e^{-3}) - (e^{3} + e^{-3})\right] + \frac{1}{6}\left[e^{6} - e^{-6}\right]$$

$$-\frac{2}{3}e^{3} - \frac{4}{3}e^{-3}$$

$$D^2 = \frac{2}{3}a^2 - a\cdot\frac{4}{9}\left[e^{3} + 2e^{-3}\right] + \frac{1}{6}\left[e^{6} - e^{-6}\right]$$

U2

$$\frac{\partial D^2}{\partial a} = \frac{4}{3}a - \frac{4}{9}\left[e^{3} + 2e^{-3}\right] = 0$$

$$\rightarrow a = \frac{1}{3}\left[e^{3} + 2e^{-3}\right] = 6.728$$

U2