# Kinematics

## Basics

Reactions and Phase Space
Units and Dimensions
Kinematics of 2-2 Reaction
3-Body Decay

## Cross Sections

Feynman Rules
Deep Inelastic ep Scattering
Drell-Yan pp Scattering

## Exercises

## Backup

Calculation of a Scattering Cross Section
Proton Structure

Kajantie, Byckling: Kinematics, 1975, about
Van Schlippe: relativistic Kinematics, 2002 – online
….

Max Klein, Lectures for Postgraduates at the University of Liverpool

November 3, 2021

## phase space

$a+b \to 1+2+\dots n$

$p_a+p_b \to p_1 + \dots \quad p_n$

$p=(E,\vec{p})$, $E^2 = M^2+\vec{p}^{\,2}$, $c=1$.

$$\left.\begin{aligned} E_a+E_b &= \sum_{i=1}^{n} E_i \\ \vec{p}_a+\vec{p}_b &= \sum_1^n \vec{p}_i \end{aligned}\right] \text{4 conditions.}$$

momentum space of $\{\vec{p}_i\}$: $3n$

phase space : $3n-4$ dim. surface

Matrixelements: $\langle \vec{p}_1 \cdots \vec{p}_n | A | \vec{p}_a \vec{p}_b \rangle = A(\vec{p}_i)$

define $T = |A(\vec{p}_i)|^2$.

total cross section $p_a+p_b \to p_1 \cdots p_n$, $(p_a+p_b)^2 = s$, $p_i^2 = m_i^2$

$$\sigma_{tot}^{(n)}(s,m_i) = \frac{1}{\phi}\cdot \underbrace{\int \prod_{i=1}^{n} \frac{d^3p_i}{2E_i}\, \delta^4\left(p_a+p_b-\Sigma p_i\right)}_{\text{phase space integral } (T=1)} T(\vec{p}_i) = \frac{I_n}{\phi}.$$

$\phi$: "Flux factor" $= \phi = 2\lambda^{1/2}(s,m_a^2,m_b^2)\underline{(2\pi)^{3n-4}}$ ← often included in $I_n$.

$$(\lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx).$$

lifetime : $\tau = \frac{1}{\sigma_{tot}^{\cdot\cdot}(M)} = \frac{\phi(M)}{I_k(M)}$. $\quad \phi(M) = 2M\cdot(2\pi)^{3k-4}$

$p \to p_1 \cdots p_k$

$$I_k(M^2) = \int \prod_1^k \frac{d^3p_i}{2E_i}\,\delta^4(p-\Sigma p_i)\, T$$

differential cross section $[a+b \to 1\dots n]$

$$\frac{d\sigma_n}{dx} = \frac{1}{\phi}\int \prod_1^n \frac{d^3p_i}{2E_i}\,\delta^4(p_a+p_b-\Sigma p_i)\,\delta(x-x(\vec{p}_i))\cdot T(\vec{p}_i).$$

distribution: normalised differential cross section

$$W(x) = \frac{1}{\sigma}\frac{d\sigma}{dx}.$$

# Units

**Convention in particle physics:**

$\hbar$ = one unit of action ($ML^2/T$), c = one unit of velocity ($L/T$)

mass (m), momentum (mc), energy ($mc^2$) are of dimension [GeV]

length (h/mc), time ($h/mc^2$) are of dimension [$GeV^{-1}$]

Conventional Mass, Length, Time Units, and Positron Charge in Terms of $\hbar = c = 1$ Energy Units

| Conversion Factor | $\hbar = c = 1$ Units | Actual Dimension |
|---|---|---|
| $1\ \text{kg} = 5.61 \times 10^{26}$ GeV | GeV | $\frac{\text{GeV}}{c^2}$ |
| $1\ \text{m} = 5.07 \times 10^{15}\ \text{GeV}^{-1}$ | $\text{GeV}^{-1}$ | $\frac{\hbar c}{\text{GeV}}$ |
| $1\ \text{sec} = 1.52 \times 10^{24}\ \text{GeV}^{-1}$ | $\text{GeV}^{-1}$ | $\frac{\hbar}{\text{GeV}}$ |
| $e = \sqrt{4\pi\alpha}$ | — | $(\hbar c)^{1/2}$ |

F.Halzen, A.Martin “Quarks and Leptons” Wiley&Sons, 1984

# Units and Dimensions

$$\hbar = 6.6\cdot 10^{-16}\ eVs$$

$$c = 3\cdot 10^{8}\ m/s$$

$$\Delta x = \frac{\hbar c}{E} = 0.2 fm \quad \text{for } E = 1 GeV.$$

$$\hookrightarrow 1m = 5\cdot 10^{15}\frac{1}{GeV}$$

$$\frac{1}{GeV^2} = 0.04\cdot 10^{-30}\,m^2 = 0.4\,mb$$

$$1\,barn = 10^{-28}\,m^2$$

$$m_e c^2 = 0.51\cdot 10^{-3}\ GeV.$$

$$\hbar c = 2\cdot 10^{-16}\,m\ GeV.$$

## Dimension Counting. Example: ey → ey Thompson Scattering

$$\sigma = \frac{2}{3}\alpha^2\cdot 4\pi R_e^2$$

$$R_e = \frac{\hbar}{m_e c}$$

$$\alpha = (4\pi)^{-1}e^2 = \frac{1}{137}$$

$$\sigma = \frac{8\pi}{3}\alpha^2\frac{\hbar^2}{m_e^2 c^2}.$$

in PP $\hbar = 1,\ c = 1$

$$\hookrightarrow \sigma = \frac{8\pi}{3}\left(\frac{\alpha}{m_e}\right)^2.$$

$$\sigma = \frac{8\pi}{3}\left(\frac{\alpha}{m_e}\right)^2\hbar^a c^b$$

$$a = +2 \qquad b = -2.$$

$$\sigma = \frac{8\pi}{3}\alpha^2\frac{(\hbar c)^2}{(m_e c^2)^2} \qquad \frac{[m\ GeV]^2}{[GeV]^2} = m^2$$

$$\sigma = \frac{8\pi}{3}\alpha^2\left(\frac{\hbar}{m_e c}\right)^2.$$

$\rightarrow \sigma = ?\ barn$

$R_e$ = Compton wavelength of the electron

## Units and Dimensions

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$\Delta x \approx \frac{\hbar c}{E} = 0.2\,\text{fm}$ for E = 1 GeV.

$\hookrightarrow 1\,\text{m} = 5 \cdot 10^{15} \frac{1}{\text{GeV}}$

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$a = +2 \quad b = -2.$

$\sigma = \frac{8\pi}{3} \alpha^2 \frac{(\hbar c)^2}{(m_e c^2)^2} \qquad \frac{[\text{m GeV}]^2}{[\text{GeV}]^2} = \text{m}^2$

$\sigma = \frac{8\pi}{3} \alpha^2 \left(\frac{\hbar}{m_e c}\right)^2.$

$(2 \; 10^{-16}\,\text{m} / 0.5 \; 10^{-3})^2 = 16 \; 10^{-26}\,\text{m}^2$

$4\pi\alpha^2 = 0.00067$

→ $4\pi\alpha^2 R_e = 1\;barn$

→ $\sigma = 2/3\;barn$

$R_e$ = Compton wavelength of the electron [often h/mc: $2.4 \; 10^{-12}$ m]

# barn

The etymology of the unit barn is whimsical: during Manhattan Project research on the atomic bomb during World War II, American physicists at Purdue University needed a secretive unit to describe the approximate cross sectional area presented by the typical nucleus ($10^{-28}$ $m^2$) and decided on "barn". This was particularly applicable because they considered this a large target for particle accelerators that needed to have direct strikes on nuclei and the American idiom "couldn't hit the broad side of a barn"[2] refers to someone whose aim is terrible. Initially they hoped the name would obscure any reference to the study of nuclear structure; eventually, the word became a standard unit in nuclear and particle physics.

Wikipedia

# Lifetime

The lifetime of the muon is given as $\tau_\mu = 192\pi^3/G_F^2 M_\mu^5$.

($M_\mu$=0.11GeV, G=1.17 $10^{-5}$ $GeV^{-2}$). Calculate $\tau_\mu$ in seconds. How large is the tau lifetime ($M_\tau$=1.78 GeV)?

**Decay width**

$$\Gamma = \frac{\hbar}{\tau}$$ **lifetime at rest**

$$\Gamma = \frac{1}{\tau}$$

$$\Gamma = \sum_{f=1}^{n} \Gamma_f$$ **n decay channels**

**Branching ratio**

$$b_f = \frac{\Gamma_f}{\Gamma} = \frac{\tau}{\tau_f}$$

$\mu^- \to \nu_\mu e^- \overline{\nu_e}$ **b=100%**

$\tau^- \to \nu_\tau e^- \overline{\nu_e}$ **b= 17%**

**Lifetime**

**long lived: > $10^{-16}$s**
Measure decay lengths
with high resolution detectors

**short lived:** determine width
of resonant state from invariant
mass distribution of decay
particle momenta → lieftime

# Lifetime

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($M_\mu$=0.11GeV, G=1.17 $10^{-5}$ $GeV^{-2}$). Calculate $\tau_\mu$ in seconds. How large is the tau lifetime ($M_\tau$=1.78 GeV)?

$\mu \rightarrow \nu_\mu \nu_e e$. This decay is nearly 100%, there is also the decay + $\gamma$ to 1.4%.
It is required to restore the dimensions as was explained in lecture 1. Here

$$\tau_\mu = \frac{192\pi^3}{G_F^2 M_\mu^5} h^1 c^0$$

should be ħ in the equation

since [ħ]=[eVs].
With the value of ħ =6.6 $10^{-16}$ eV s one finds [ħ is given in the PDG list of constants]

**$\tau_\mu$ = 2.14 $10^{-6}$ s** (the PDG value is 2.19) in a then straightforward calculation.

$\tau_\tau$

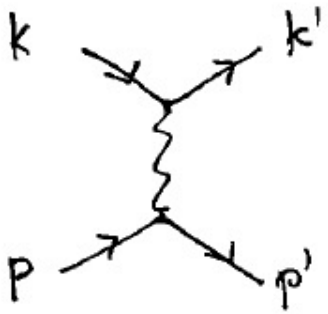
?

$\tau_\tau$

# Mandelstam Variables

s, t, u

Stanley Mandelstam 1928-2016.




ep → ep, similar for pp → pp etc.

$$s = (k+p)^2 = (k'+p')^2 = (\text{cms energy})^2$$

$$s = k^2 + p^2 + 2k_0 p_0 - 2\vec{k}\vec{p}$$

$k^2 = M_e^2$, $p^2 = M_p^2 = p_0^2 - \vec{p}^{\,2}$

large momenta: $k_0 = E_e = |\vec{k}|$, $p_0 \approx E_p \approx |\vec{p}|$

$$s \approx 2E_e E_p (1 - \cos\vartheta_{\vec{k}\vec{p}})$$

$\cos\vartheta_{\vec{k}\vec{p}} = -1$ for head on collisions

$$s \approx 4E_e E_p$$

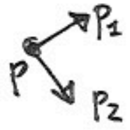
$$t = (k-k')^2 = (p-p')^2 = (\text{4 momentum transfer})^2$$

$$u = (p-k')^2 = (k-p')^2$$

$$s+t+u = (k+p)^2 + (k-k')^2 + (p-k')^2 = f(m_e, m_p) = ?$$

$s+t+u =$ ? for identical masses

# Two-body Decay




$P = (M, \vec{P})$ $\quad$ at rest.

$\vec{P} = 0$

$p_1 = (E_1, \vec{p}_1) \quad p_2 = (E_2, \vec{p}_2).$

$P = p_1 + p_2 \rightarrow \vec{p}_1 = -\vec{p}_2 = \vec{p}$ back to back.

$E_1 + E_2 = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} = M.$

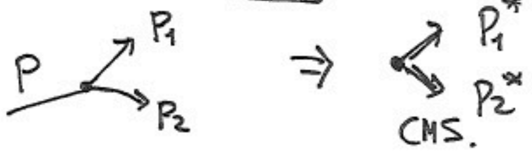
$\hookrightarrow p = \frac{1}{2M}\sqrt{[M^2 - (m_1 - m_2)^2][M^2 - (m_1+m_2)^2]}$

$M \geq m_1 + m_2.$ $\quad$ if $m_1 = m_2$:

$$p = \frac{1}{2}\sqrt{M^2 - 4m^2}$$

$E_1 = \frac{1}{2M}(M^2 + m_1^2 - m_2^2) = \frac{M}{2}$ for $m_1 = m_2$.

in flight.




$P = (E/c, \vec{P})$

$= (E/c, P_\perp, P_z).$

Lorentz transformation: $\vec{P} = M \cdot \vec{v}$ , use z-axis.

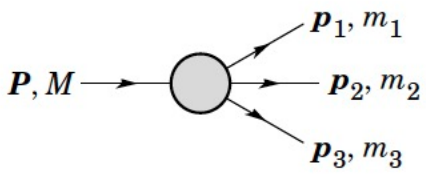
$$\begin{pmatrix} ct^* \\ z^* \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \cdot \begin{pmatrix} ct \\ z \end{pmatrix}. \Rightarrow \begin{pmatrix} E^*/c \\ p_z^* \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}}_{L} \begin{pmatrix} E/c \\ p_z \end{pmatrix}, \ \beta = \frac{v}{c}$$

$$\begin{pmatrix} E/c \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \frac{1}{\det(L)} \cdot \begin{pmatrix} E^*/c \\ p_z^* \end{pmatrix}. \quad \det(L) = \gamma^2 - \beta^2\gamma^2 = 1 \,!$$

$\rightarrow E = \gamma E^* + v\gamma p_z^* = \gamma(E^* + v p_z^*)$

$p_z = \gamma p_z^* + \gamma v E^* = \gamma(p_z^* + v E^*)$

Lab. $\qquad$ CMS.




# Dalitz Plot (kinematics of 3-body decay)

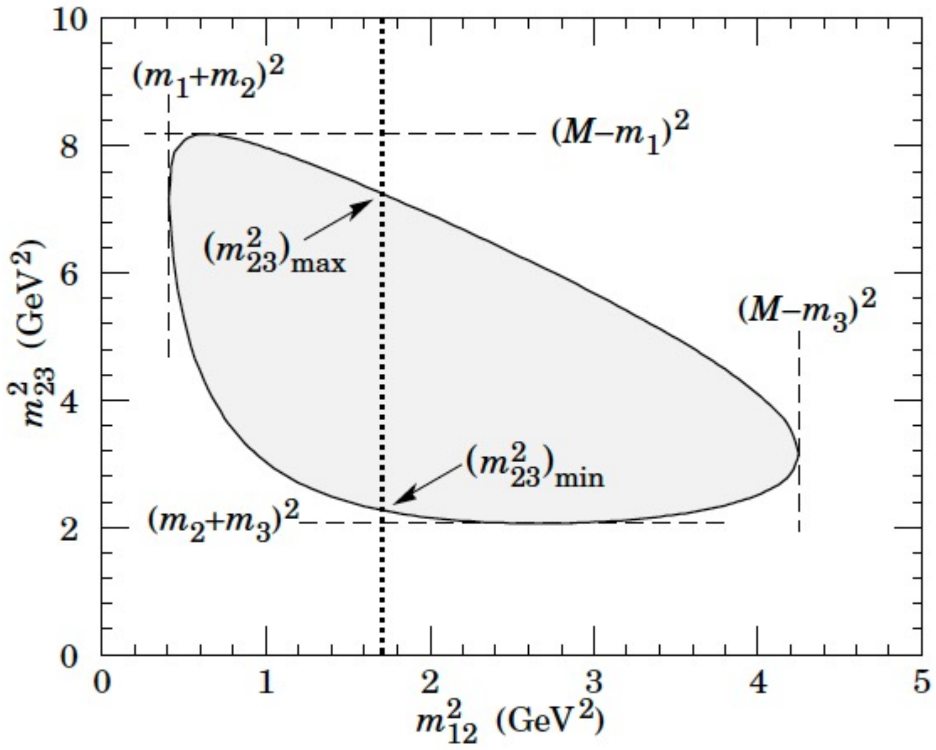
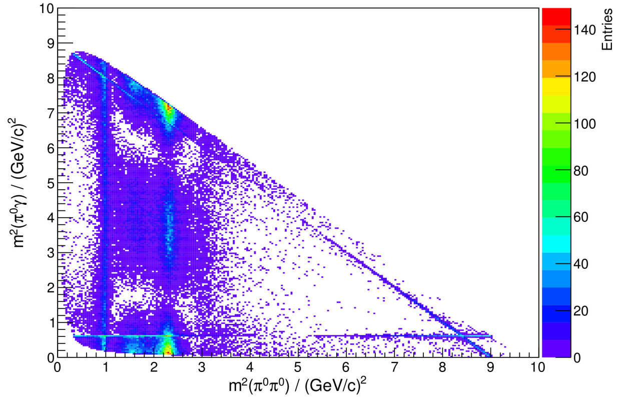



**Figure 46.3:** Dalitz plot for a three-body final state. In this example, the state is $\pi^+\overline{K}^0 p$ at 3 GeV. Four-momentum conservation restricts events to the shaded region.

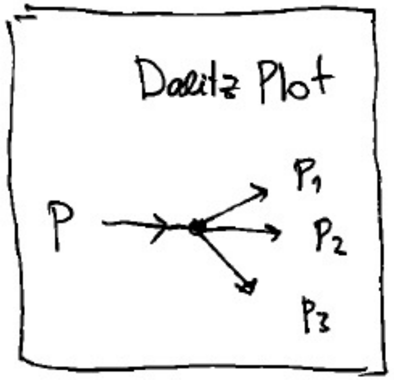
$J/\psi \rightarrow \pi^0\pi^0\gamma$




Richard Henry Dalitz (1925-2006)

Cambridge
Bristol
Birmingham
Cornell
Chicago
Oxford

# Dalitz Plot




$$s = P^2$$

$$s_1 = (P-p_1)^2 = (p_2+p_3)^2$$

$$s_2 = (p_1+p_3)^2$$

$$s_3 = (p_1+p_2)^2.$$

$$s_1+s_2+s_3 = M^2+m_1^2+m_2^2+m_3^2.$$

if $P=(M,\vec{0})$ : $s_1 = M^2+m_1^2-2mE_1$ — Restframe of Parent

$$E_1 = \sqrt{m_1^2+p_1^2} \geq m_1$$

$$\max(s_1) = (M-m_1)^2.$$

if $(\vec p_2+\vec p_3)=0$ Jackson Frame $j$: $s_1 = (E_2^j+E_3^j)^2 \geq (m_2+m_3)^2$ — Restframe of 2+3 to find $\min(s_1)$

$s_1 \in [(m_2+m_3)^2, (M-m_1)^2]$, similar for $s_2, s_3$

Jackson Frame $R_{23}$: $\vec p_2+\vec p_3 = 0$ . ↷ $\vec P = \vec p_1$

$$s_1 = (P^j-p_1^j)^2 = (E^j-E_1^j)^2 = \left(\sqrt{M^2+\vec p_1^{j\,2}} - \sqrt{m_1^2+\vec p_1^{j\,2}}\right)^2$$

↪ $$p_1^{j\,2} = \frac{1}{4s_1}\cdot[s_1-(M-m_1)^2][s_1-(M+m_1)^2] = \frac{1}{4s_1}\lambda(s_1,M^2,m_1^2)$$

$$\lambda(x,y,z) := x^2+y^2+z^2-2xy-2yz-2zx.$$

also $s_1 = (p_2+p_3)^2 = (p_2^j+p_3^j)^2 = (E_2^j+E_3^j)^2$     $s_1 = m_{23}^2$

↳ $$p_2^{j\,2} = p_3^{j\,2} = \frac{1}{4s}\lambda(s_1,m_2^2,m_3^2).$$

$$s_2 = (p_1+p_3)^2 = m_1^2+m_3^2+2E_1^jE_3^j-2|\vec p_1^j||\vec p_3^j|\cos\alpha_{13}$$

$s_{2+} = \max(s_2)$: $\alpha_{13} = \pi$     $\min(s_2)$: $\alpha_{13}=0$ $= s_{2-}$

$$E_1^j = \frac{1}{2\sqrt{s_1}}(s-s_1-m_1^2)$$

$$E_3^j = \frac{1}{2\sqrt{s_1}}(s_1+m_3^2-m_2^2)$$

# Dalitz Plot

$$s_{2\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[(s - s_1 - m_1^2)(s_1 - m_2^2 + m_3^2) \pm \lambda^{1/2}(s_1, s, m_1^2)\cdot\lambda^{1/2}(s_1, M_2^2, m_3^2)\right].$$

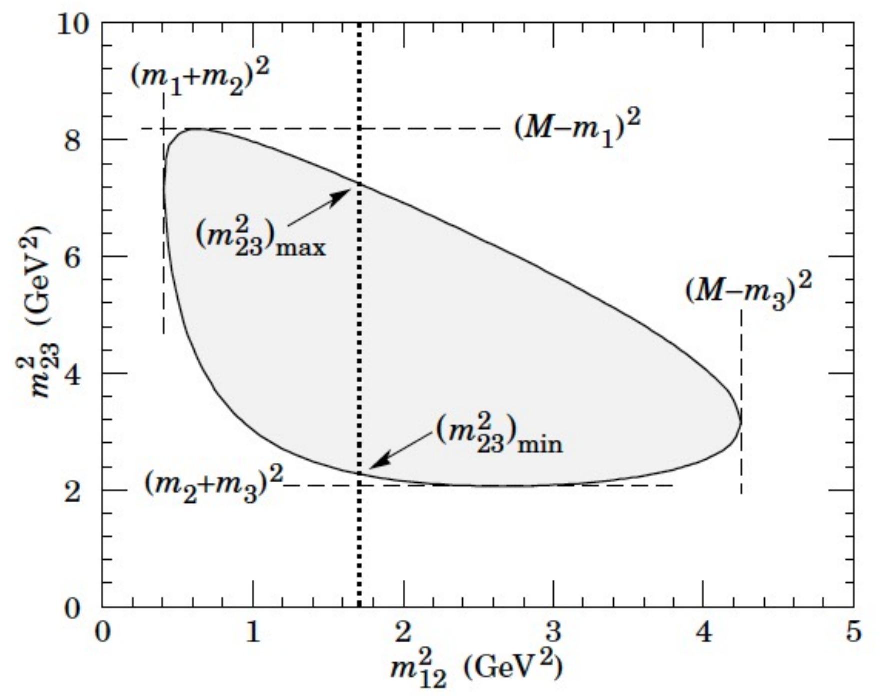



**Figure 46.3:** Dalitz plot for a three-body final state. In this example, the state is $\pi^+\overline{K}^0 p$ at 3 GeV. Four-momentum conservation restricts events to the shaded region.

This formula provides the maximum (minimum) of $s_2$ in the $s_2$-$s_1$ plane.
Similar for $s_1$-$s_3$ plane which is illustrated left

$$s_1 = (p_2+p_3)^2 = m_{23}^2 \quad !$$

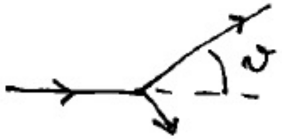
$$|\vec{p}_1|_{max} = \frac{1}{2M}\sqrt{\left[M^2-(m_1+m_2+m_3)^2\right]\left[M^2-(m_2+m_3-m_1)^2\right]}$$

max of $|\vec{p}_1|$ in the rest frame of the mother particle.

# Example: Calculation of Elastic ep Scattering Kinematics


$$E' = f(E, M, \vartheta)$$

E = E(e)
M= M(p)




$$(p_e + p_p)^2 = \cancel{p_e^2} + 2 p_e p_p + p_p^2$$

$$p_e = (E, \vec{p}_e),\ p_e^2 = 0:\ E = |\vec{p}_e|$$

$$p_p = (p_0, \vec{p}) = (M, \vec{0}).$$

$$2 p_e p_p = 2EM.$$

$$(p_e + p_p)^2 = 2EM + M^2.$$

$$(p_e' + p_p')^2 = 2E'E_p' - 2\vec{p}_e'\vec{p}_p' + \cancel{M^2} = 2EM + \cancel{M^2} = (p_e + p_p)^2$$

$$EM = E'E_p' - \cancel{2}\vec{p}_e'\vec{p}_p'$$

- energy conservation: $E_p' = E + M - E'$
- 3 momentum conservation: $\vec{p}_p' = \vec{p}_e - \vec{p}_e'$

$$EM = E'(E + M - \cancel{E'}) + \cancel{(\vec{p}_e')^2} - |\vec{p}_e||\vec{p}_e'|\cos\vartheta$$

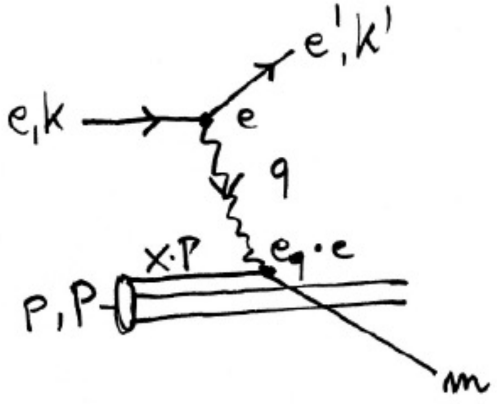
$$E'^2 \qquad EE'$$

$$EM = E'(E + M - E\cos\vartheta)$$

$$E' = E \cdot \frac{1}{1 + \frac{E}{M}(1 - \cos\vartheta)}$$

Application:
Search for Dark Matter as WIMPs generate recoils

# Deep Inelastic ep Scattering




"fixed target":

$P = (M_p, 0,0,0)$ Proton at rest

$2Pq = 2M_p(E - E')$ Energy transfer

$= 2M_p E \cdot \frac{\nu}{E} \equiv s \cdot y$ y=relative E transfer = inelasticity

$Q^2 = sxy \le s$

$s = 2M_p E$ cms energy squared

$q = (k - k')$ 4-momentum transfer

$(xP + q)^2 = m^2, P^2 = M_p^2$ Conservation of 4-momentum

$Q^2 = -q^2 > 0$

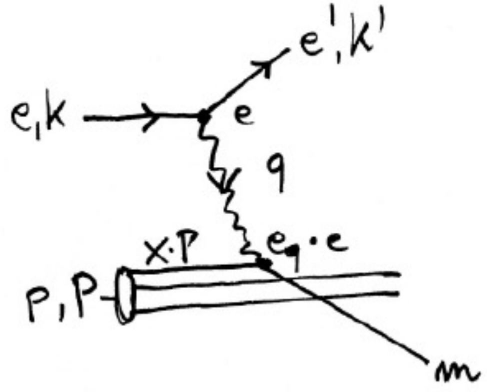
$if: Q^2 >> x^2 M_p^2, m^2:$ Deep inelastic scattering

$q^2 + 2xPq = 0:$

$x = \frac{Q^2}{2Pq}$ Bjorken x

Calculate the cms energy squared for a head-on ep collider of beam energies $E_e$, $E_p$

# Deep Inelastic ep Scattering




$q = (k - k')$ — 4-momentum transfer

$(xP + q)^2 = m^2, P^2 = M_p^2$ — Conservation of 4-momentum

$Q^2 = -q^2 > 0$

$if: Q^2 >> x^2 M_p^2, m^2:$ — Deep inelastic scattering

$q^2 + 2xPq = 0:$

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"fixed target":

$P = (M_p, 0,0,0)$ — Proton at rest

$2Pq = 2M_p(E - E')$ — Energy transfer

$= 2M_p E \cdot \frac{\nu}{E} \equiv s \cdot y$ — y=relative E transfer = inelasticity

$Q^2 = sxy \leq s$

$s = 2M_p E$ — cms energy squared

$s = (k + P)^2 = me^2 + Mp^2 + 2kP$
$\approx 2(EeEp - |\vec{pe}||\vec{P}|\cos(\vartheta) = 4E_e E_p$

Note that **the 3-momentum in absolute equals the energy if the mass is negligible**

LHeC: arXiv:2007.14491

Example: LHeC: s=4x60x7000 $\mathrm{GeV}^2$ → an equivalent fixed target electron beam would need to have 900 TeV
Quarks were discovered with a 2mile LINAC of 20 GeV. The LHeC equivalent would be 90 000 miles long.

# Quantum Electrodynamics [QED]

$$L_{QED} = \overline{\Psi} D \Psi + m \overline{\Psi} \Psi + (DA)^2 + eA \overline{\Psi} \Psi$$

e propagator e mass photon interaction

virtual particles: quantum theory

perturbative QED

**Diagrams**
**Rules**
**Integrals**

Lamb shift: 1947

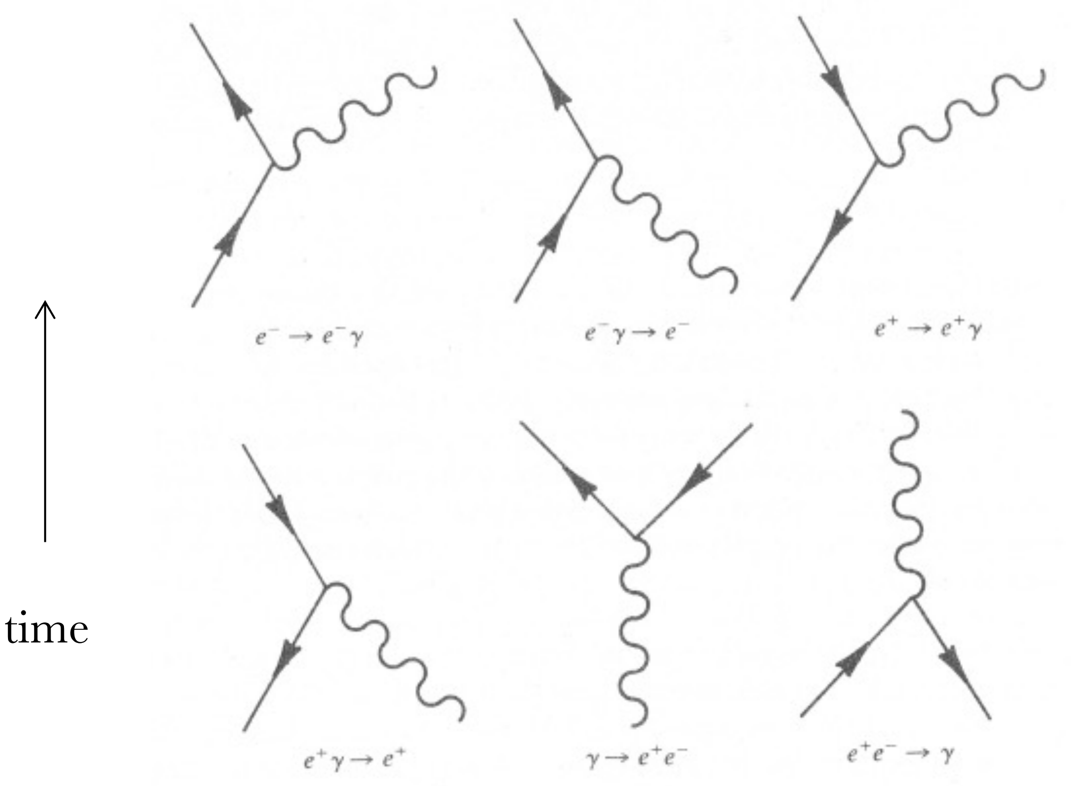
Renormalisation

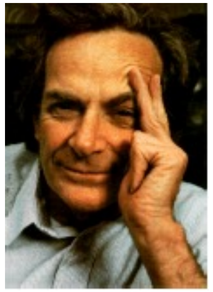
"QED and the men who made it", by S.Schweber, Princeton 1994

**Sin-Itiro Tomonaga, Julian Schwinger, Richard Feynman: Nobel 1965**

Classical mechanics: Equations of motion from Lagrange equations L=T-V
QED – a Lagrangian, renormalisable gauge field theory

# Feynman Diagrams



cf Feynman lectures
(video tapes, New Zealand)
An introduction to QED by
the master, you can listen to

Electron is a line, incoming/outgoing
Photon is a wiggly line
Emission of photon determines a vertex.
Conservation of charge and energy-momentum at the vertex
(yields δ-functions for the sum of 4-momenta)
Probability of this emission is proportional to charge $e=\sqrt{(4\pi\alpha)}$

In QED, as in other quantum field theories, we can use the little pictures invented by my colleague Richard Feynman, which are supposed to give the illusion of understanding what is going on in quantum field theory.
M.Gell-Mann

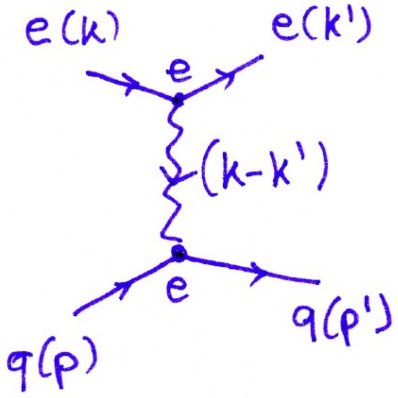
cited in "Particles and Nuclei", B.Povh et al.

# Feynman Rules

**TABLE 6.2**
Feynman Rules for $-i\mathfrak{M}$

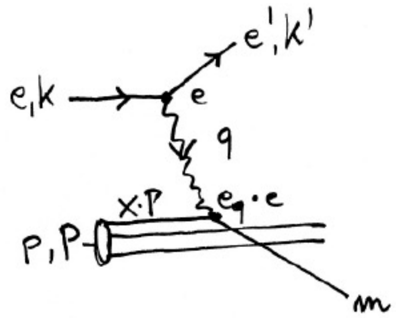
| | Multiplicative Factor |
|---|---|
| • **External Lines** | |
| Spin 0 boson (or antiboson) | 1 |
| Spin ½ fermion (in, out) | $u, \bar{u}$ |
| antifermion (in, out) | $\bar{v}, v$ |
| Spin 1 photon (in, out) | $\varepsilon_\mu, \varepsilon^*_\mu$ |
| • **Internal Lines—Propagators (need $+i\varepsilon$ prescription)** | |
| Spin 0 boson | $\frac{i}{p^2 - m^2}$ |
| Spin ½ fermion | $\frac{i(\not{p} + m)}{p^2 - m^2}$ |
| Massive spin 1 boson | $\frac{-i(g_{\mu\nu} - p_\mu p_\nu / M^2)}{p^2 - M^2}$ |
| Massless spin 1 photon (Feynman gauge) | $\frac{-ig_{\mu\nu}}{p^2}$ |
| • **Vertex Factors** ($p$, $p'$) | |
| Photon—spin 0 (charge $-e$) | $ie(p + p')^\mu$ |
| Photon—spin ½ (charge $-e$) | $ie\gamma^\mu$ |

$$T \propto u_e(k)\gamma_\mu \overline{u_e}(k') \cdot \frac{e^2}{(k-k')^2} \cdot u_q(p)\gamma^\mu \overline{u_q}(p')$$




Feynman diagrams lead to straightforward calculation of scattering amplitudes. This requires also to sum over incoming and average over final state spin states. The cross section is obtained from the square of the (complex) amplitude (TT*) taking into account phase space factors.

# DIS Cross Section




Electroweak NC interactions in inclusive $e^{\pm}p$ DIS are mediated by exchange of a virtual photon ($\gamma$) or a $Z$ boson in the $t$-channel, while CC DIS is mediated exclusively by $W$-boson exchange as a purely *weak* process. Inclusive NC DIS cross sections are expressed in terms of generalised structure functions $\tilde{F}_2^{\pm}$, $x\tilde{F}_3^{\pm}$ and $\tilde{F}_{\mathrm{L}}^{\pm}$ at EW leading order (LO) as

Photon exchange

~ $1/Q^4$ $F_2$

$$\frac{d^2\sigma^{\mathrm{NC}}(e^{\pm}p)}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}\left[Y_+\tilde{F}_2^{\pm}(x,Q^2) \mp Y_- x\tilde{F}_3^{\pm}(x,Q^2) - y^2\tilde{F}_L^{\pm}(x,Q^2)\right] , \quad (5.1)$$

where $\alpha$ denotes the fine structure constant. The terms $Y_{\pm} = 1 \pm (1-y)^2$, with $y = Q^2/sx$, describe the helicity dependence of the process. The generalised structure functions are separated into contributions from pure $\gamma$- and $Z$-exchange and their interference [96, 134]:

Neutral Current:

$$\tilde{F}_2^{\pm} = F_2 - (g_V^e \pm P_e g_A^e)\kappa_Z F_2^{\gamma Z} + [(g_V^e g_V^e + g_A^e g_A^e) \pm 2P_e g_V^e g_A^e]\,\kappa_Z^2 F_2^Z , \quad (5.2)$$

$$\tilde{F}_3^{\pm} = \; -(g_A^e \pm P_e g_V^e)\kappa_Z F_3^{\gamma Z} + [2g_V^e g_A^e \pm P_e(g_V^e g_V^e + g_A^e g_A^e)]\,\kappa_Z^2 F_3^Z . \quad (5.3)$$

Photon and Z exchange

Similar expressions hold for $\tilde{F}_L$. In the naive quark-parton model, which corresponds to the LO QCD approximation, the structure functions are calculated as

Parton Distributions (PDFs)
$xq(x,Q^2)$

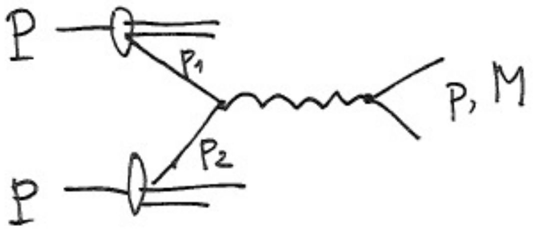
$$\left[F_2, F_2^{\gamma Z}, F_2^Z\right] = x\sum_q \left[Q_q^2, 2Q_q g_V^q, g_V^q g_V^q + g_A^q g_A^q\right]\{q+\bar{q}\} , \quad (5.4)$$

$$x\left[F_3^{\gamma Z}, F_3^Z\right] = x\sum_q \left[2Q_q g_A^q, 2g_V^q g_A^q\right]\{q-\bar{q}\} , \quad (5.5)$$

x dependence from experiment

representing two independent combinations of the quark and anti-quark momentum distributions, $xq$ and $x\bar{q}$. In Eq. (5.3), the quantities $g_V^f$ and $g_A^f$ stand for the vector and axial-vector couplings of a fermion ($f = e$ or $f = q$ for electron or quark) to the $Z$ boson, and the coefficient $\kappa_Z$ accounts for the $Z$-boson propagator including the normalisation of the weak couplings. Both

# Drell-Yan Scattering




$$x_1 \cdot x_2 = \frac{M^2}{s} = \tau \quad , \quad s = 4E_p^2$$

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \left[ \frac{E_1 + E_2 + (E_1 - E_2)}{E_1 + E_2 - (E_1 - E_2)} \right]$$

$$y = \frac{1}{2} \ln \left[ \frac{x_1 + x_2 + x_1 - x_2}{x_1 + x_2 - x_1 + x_2} \right] \qquad y = \frac{1}{2} \ln \left( \frac{x_1}{x_2} \right)$$

$$p_1 = (E_1, \vec{0}_\perp, E_1)$$

$$p_2 = (E_2, \vec{0}_\perp, E_2)$$

$$E_{1,2} = x_{1,2} \cdot E_p$$

$$(p_1 + p_2)^2 = p^2 = M^2$$

$$2E_1E_2 - 2E_1E_2 \cos\vartheta$$

$$= 4E_1E_2 = 4x_1x_2 \cdot E_p^2 = x_1x_2 \cdot s = M^2$$

$$x_1 \cdot x_2 = \tau \quad , \quad e^y = \sqrt{\frac{x_1}{x_2}} = x_1 \cdot \sqrt{\tau}$$

$$x_1 = \frac{M}{\sqrt{s}} e^y \quad . \quad x_2 = \frac{M}{\sqrt{s}} e^{-y}$$

$y = 0$ : central production , $M = M_{Higgs} = 125$ GeV

$\sqrt{s} = 13$ TeV (LHC in 2016) : $x_1 = x_2 \approx 0.01$

$\eta$ or $y \simeq 2.5$ : $\vartheta \simeq 10°$ : $x \approx 10^{-3}$ to $x \approx 0.1$

# Rapidity

$$E = \gamma m c^2 \quad [= \gamma m]$$

$$|\vec{p}| = \beta\gamma m c \quad [= \beta\gamma m]$$

$$\beta = \frac{v}{c}. \qquad y \doteq \text{artanh}(\beta)$$

$$\cosh y = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

$$\sinh y = \frac{\beta}{\sqrt{1-\beta^2}} = \beta\gamma.$$

$$\hookrightarrow E = m\cdot\cosh y, \quad |\vec{p}| = m\sinh y$$

$$\left\{\text{artanh}\, x = \frac{1}{2}\ln\frac{1+x}{1-x}\right\}$$

$$\frac{|\vec{p}|}{E} = \tanh y \rightarrow y = \frac{1}{2}\ln\frac{E+|\vec{p}|}{E-|\vec{p}|}$$

L transformation

$$y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \frac{1}{2}\ln\frac{\gamma E^* + \gamma\beta p_z^* + \gamma\beta E^* + \gamma p_z^*}{\gamma E^* + \gamma\beta p_z^* - \gamma\beta E^* - \gamma p_z^*}$$

$$= \frac{1}{2}\ln\left[\left(\frac{E^*+p_z^*}{E^*-p_z^*}\right)\left(\frac{1+\beta}{1-\beta}\right)\right] = y^* + y_{CMS}.$$

CMS: $E_{CMS} = \sqrt{s}$, $p_z = 0$ ; $E^* = \gamma\sqrt{s}$, $p_z^* = \beta\gamma\sqrt{s} \Rightarrow y = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}.$

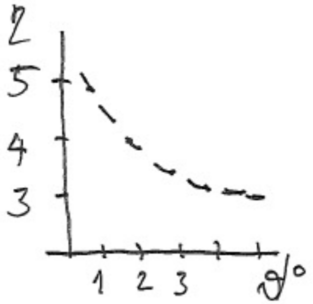
pseudorapidity

Mass $\simeq 0$ : $E = E$, $p_z = E\cdot\cos\vartheta$

$$\eta = y(m=0) = \frac{1}{2}\ln\left(\frac{1+\cos\vartheta}{1-\cos\vartheta}\right).$$

$$\underbrace{\phantom{xx}}_{=\left(\frac{1}{\tan\vartheta/2}\right)^2}$$

$$\eta = -\ln\tan\frac{\vartheta}{2}$$



# Drell Yan Cross Section

To leading order, the double differential Drell-Yan scattering cross section [3] for the neutral current (NC) reaction $pp \to (Z/\gamma)X \to e^+e^-X$ and the charged current (CC) reaction $pp \to W^{\pm}X \to e^{\pm}\nu_e(\overline{\nu}_e)X$, can be written as

$$\frac{d^2\sigma}{dMdy} = \frac{4\pi\alpha^2(M)}{9} \cdot 2M \cdot P(M) \cdot \Phi(x_1, x_2, M^2) \,. \qquad (1)$$

Photon exchange

$\sim 1/M^4 \;\; F_y$

Here $M$ is the mass of the $e^+e^-$ and $e^+\nu$ and $e^-\bar{\nu}$ systems for the NC and CC process, respectively, and $y$ is the boson rapidity. The cross section implicitly depends on the Bjorken $x$ values of the incoming quark $q$ and its anti-quark $\overline{q}$, which are related to the rapidity $y$ as

$$x_1 = \sqrt{\tau}e^{y} \quad x_2 = \sqrt{\tau}e^{-y} \quad \tau = \frac{M^2}{s} \quad s = 4E_p^2. \qquad (2)$$

For the NC process, the cross section is a sum of a contribution from photon and $Z$ exchange as well as an interference of them. In the case of photon exchange, the propagator term $P(M)$ and the parton distribution term $\Phi$ are given by

$$P_\gamma(M) = \frac{1}{M^4} \qquad \Phi_\gamma = \sum_q e_q^2 F_{q\overline{q}} \qquad (3)$$

$$F_{q\overline{q}} = x_1 x_2 \cdot [q(x_1, M^2)\overline{q}(x_2, M^2) + \overline{q}(x_1, M^2)q(x_2, M^2)]. \qquad (4)$$

ATLAS Internal Note 2010

# Drell Yan Cross Section

The corresponding formulae for the $\gamma Z$ interference term read as

$$P_{\gamma Z} = \frac{\kappa_Z v_e (M^2 - M_Z^2)}{M^2[(M^2 - M_Z^2)^2 + (\Gamma_Z M_Z)^2]} \qquad \Phi_{\gamma Z} = \sum_q 2 e_q v_q F_{q\bar{q}} \tag{5}$$

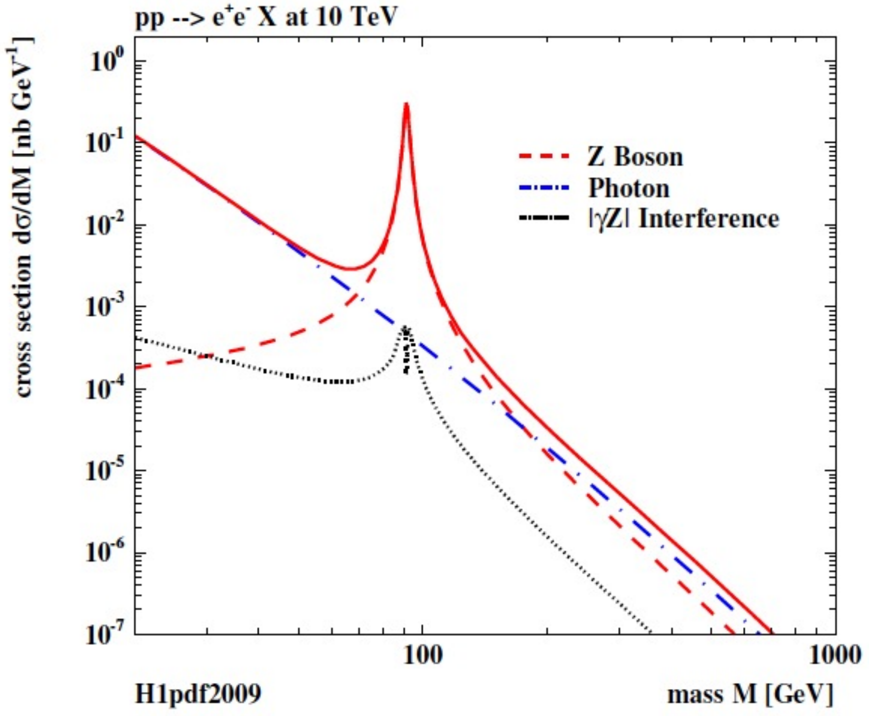
$$v_f = I_3^f - e_f \sin^2\Theta, \;\; a_f = I_3^f \;\; [f = e, q] \qquad \kappa_z = \frac{1}{4\sin^2\Theta\cos^2\Theta} \qquad \cos\Theta = \frac{M_W}{M_Z}, \tag{6}$$

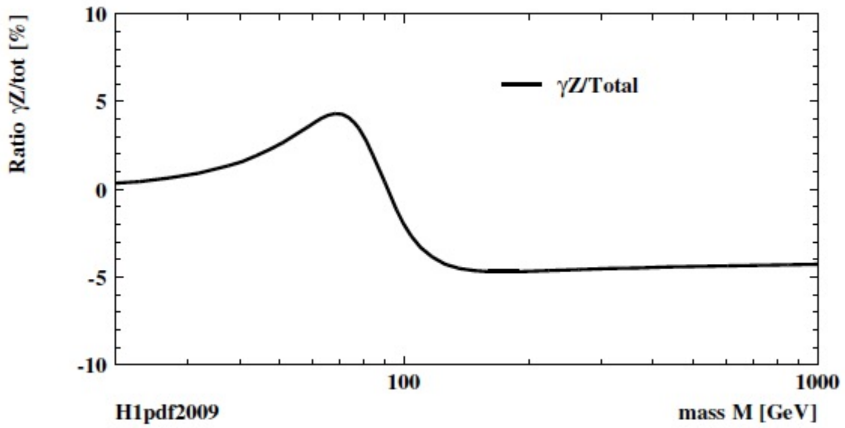
where in the weight of the parton distributions one electric charge $e_q$ is replaced by twice the neutral current vector coupling $v_q$. The interference contribution is proportional to the vector coupling of the electron $v_e$. Because of $I_3^e = -1/2$ and $\sin^2\Theta$ being close to $1/4$, $v_e$ is small and thus the $\gamma Z$ cross section part is also small. One also sees in Eq. 5 that the interference cross section contribution changes sign from positive to negative as the mass increases and passes $M_Z$. The neutral current Drell-Yan cross section formulae are completed by the expressions of $P$ and $\Phi$ for the pure $Z$ exchange part, in which the vector and axial-vector couplings enter as sums $v^2 + a^2$ as

$$P_Z = \frac{\kappa_Z^2 (v_e^2 + a_e^2)}{(M^2 - M_Z^2)^2 + (\Gamma_Z M_Z)^2} \qquad \Phi_Z = \sum_q (v_q^2 + a_q^2) F_{q\bar{q}}. \tag{7}$$

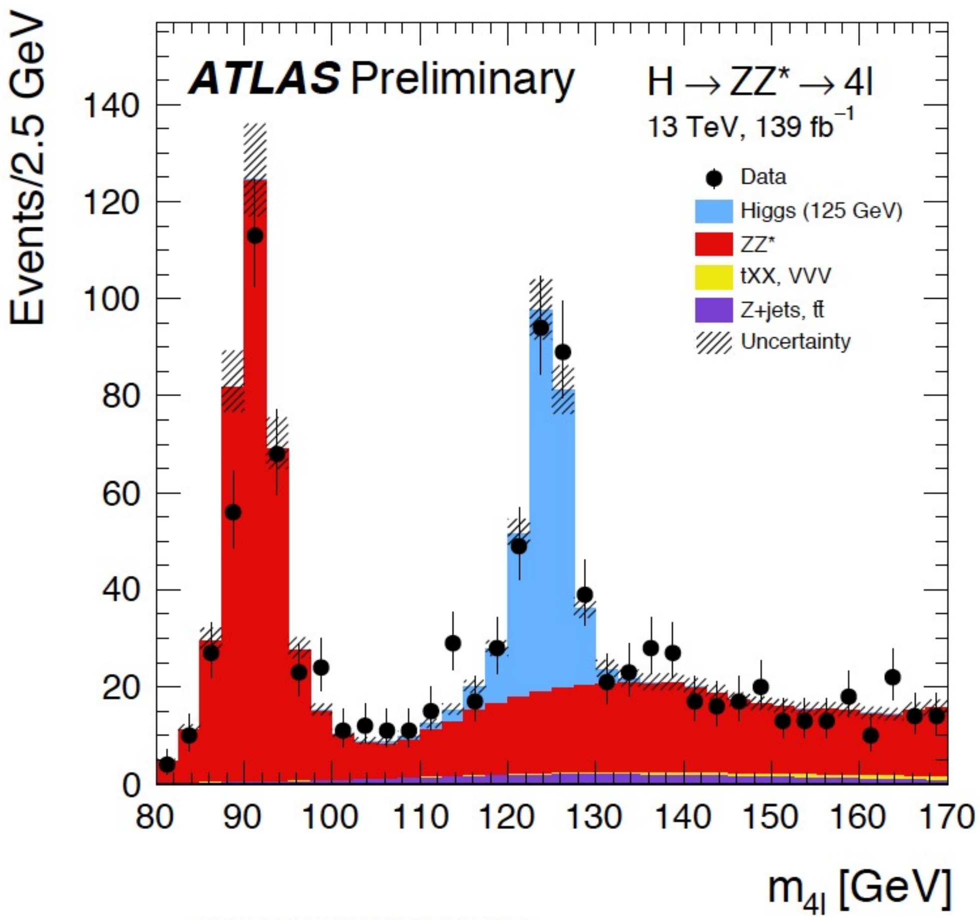
# Drell Yan

pp → ee X vs M(ee)




pp → 4l X

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# Exercises

# Decay

In an experiment a particle decay at rest is observed into a muon and a neutrino. The mass of the muon is known to be $M_{\mu}$=106 MeV and the kinetic energy of the muon is measured to be T=4.4 MeV. Determine the mass of the parent particle and identify it with a known particle.

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| | |
|---|---|
| $p_\pi^2 = M_\pi^2 = E_\pi^2 - k_\pi^2$ | 4 vector relation |
| $\rightarrow M_\pi = E_\pi$ | Pion decays at rest |
| $E_\pi = E_\mu + E_\nu$ | Energies add up |
| $E_\mu = M_\mu + T_\mu = 110.4 MeV$ | Kinetic energy T from track |
| $k_\mu = \sqrt{E_\mu^2 - M_\mu^2} = 31 MeV$ | 3-momentum of muon |
| $k_\pi = 0$ | 3-momentum of pion is zero |
| $\rightarrow k_\nu = k_\mu = 31 MeV = E_\nu$ | Neutrino is massless (about) |
| $M_\pi = E_\mu + E_\nu = 141.4 MeV$ | Energies add up to pion mass |

# Lifetime

The lifetime of the muon is given as $\tau_\mu = 192\pi^3/G_F^2 M_\mu^5$.
($M_\mu$=0.11GeV, G=1.17 $10^{-5}$ $GeV^{-2}$). Calculate $\tau_\mu$ in seconds. How large is the tau lifetime ($M_\tau$=1.78 GeV)?

$\mu \to \nu_\mu \nu_e e$. This decay is nearly 100%, there is also the decay + $\gamma$ to 1.4%.
It is required to restore the dimensions as was explained in lecture 1. Here

$$\tau_\mu = \frac{192\pi^3}{G_F^2 M_\mu^5} h^1 c^0$$

should be ħ in the equation

since [ħ]=[eVs].
With the value of ħ =6.6 $10^{-16}$ eV s one finds [ħ is given in the PDG list of constants]

$\boldsymbol{\tau_\mu = 2.14 \; 10^{-6}}$ **s** (the PDG value is 2.19) in a then straightforward calculation.

$\tau_\tau$

?

$\tau_\tau$

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$$\tau_\tau = \tau_\mu \cdot \frac{M_\mu^5}{M_\tau^5} \cdot br(\nu_\tau \nu_e e) \approx 2.14 \cdot 10^{-6} \cdot 7.5 \cdot 10^{-7} \cdot 0.17 s$$

$$\tau_\tau \approx 2.7 \cdot 10^{-13} s$$

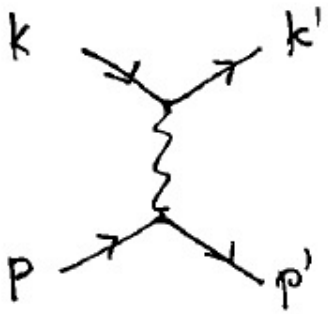
(the PDG value is 2.9
Emphasize that tau decays only to 17% into this channel,
also 17% into the muon channel, point to hadron decays.)

$$\tau_\tau = br\tau\tau(f)$$
$$\tau_\mu = br\tau\mu(f)$$
$$\tau_\tau = \tau\mu br(\tau \rightarrow f)\tau\tau(f)/\tau\mu(f)$$
$$= \tau_\mu br(\tau \rightarrow f) {M_\mu}^5/{M_\tau}^5$$

# Mandelstam Variables

s, t, u

Stanley Mandelstam 1928-2016.




ep → ep, similar for pp → pp etc.

$$s = (k+p)^2 = (k'+p')^2 = (\text{cms energy})^2$$

$$s = k^2 + p^2 + 2k_0p_0 - 2\vec{k}\vec{p}$$

$k^2 = M_e^2$, $p^2 = M_p^2 = p_0^2 - \vec{p}^{\,2}$

large momenta: $k_0 = E_e = |\vec{k}|$, $p_0 = E_p \approx |\vec{p}|$

$$s \approx 2E_eE_p(1 - \cos\vartheta_{\vec{e}\vec{p}})$$

$\cos\vartheta = -1$ for head on collisions

$$s \approx 4E_eE_p$$

$$t = (k-k')^2 = (p-p')^2 = (4\text{ momentum transfer})^2$$

$$u = (p-k')^2 = (k-p')^2$$

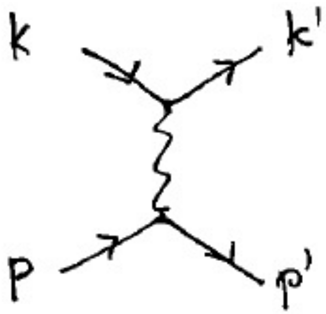
$$s+t+u = (k+p)^2 + (k-k')^2 + (p-k')^2 = f(m_e, m_p) = ?$$

$s+t+u =$ ? for identical masses

# Mandelstam Variables

s, t, u

Stanley Mandelstam 1928-2016.

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$k^2 = M_e^2$, $p^2 = M_p^2 = p_0^2 - \vec{p}^{\,2}$

large momenta: $k_0 = E_e = |\vec{k}|$, $p_0 = E_p \approx |\vec{p}|$

$$s \simeq 2E_e E_p (1 - \underbrace{\cos\vartheta_{\vec{e},\vec{p}}}_{=-1 \text{ for head on collisions}})$$

$$s \simeq 4E_e E_p$$

$$t = (k-k')^2 = (p-p')^2 = (\text{4 momentum transfer})^2$$

$$u = (p-k')^2 = (k-p')^2$$

$$s+t+u = (k+p)^2 + (k-k')^2 + (p-k')^2 = k^2 + p^2 + k'^2 + \underbrace{(k+p-k')^2}_{p'^2} = 2M_e^2 + 2M_p^2$$

$s+t+u = 4m^2$ for identical masses

# An Exercise

A Higgs particle $H$ may decay into two photons ($H \to \gamma\gamma$) with energies $E_1$ and $E_2$ and an angle of $\alpha$ between the two photons, all measured in the laboratory frame. Calculate the mass $m_H$ of the Higgs particle as a function of the measured values of $E_1$, $E_2$ and $\alpha$. Which mass in GeV do you obtain for values of $E_1 = 100\,\text{GeV}$, $E_2 = 225\,\text{GeV}$ and $\alpha = 49.2°$?

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$$p_{\gamma 1} = (E_1, \vec{p}_1) \quad \text{and} \quad p_{\gamma 2} = (E_2, \vec{p}_2)$$

$$\begin{aligned} m_H^2 &= (p_{\gamma 1} + p_{\gamma 2})^2 \\ &= (p_{\gamma 1})^2 + (p_{\gamma 2})^2 + 2 p_{\gamma 1} p_{\gamma 2} \\ &= 0 + 0 + 2\,(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2) \\ &= 2 E_1 E_2 (1 - \cos\alpha) \end{aligned}$$

$$\begin{aligned} m_H &= \sqrt{2 \cdot 225\text{GeV} \cdot 100\text{GeV}(1 - \cos(49.2/180 \cdot \pi)} \\ &\approx 124.9\text{GeV} \end{aligned}$$

# Kinematics

Good luck to your further research, PhDs and beyond. Thanks.

# Calculation of a Cross Section (Born)

backup

# Four-Vectors, Energy and Momentum Conservation

- Invariance of $(ct)^2 - z^2$ suggests to combine $(ct, z)$
- This generalises a *three vector* $\vec{x}$ to a *four vector* with the notation

$$x_\mu = (x_0, x_1, x_2, x_3)$$
$$x_\mu = (ct, x, y, z) = (ct, \vec{x})$$

- Norm (squared) of a three vector $\vec{x}$ is given by the scalar product: $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x} = \vec{x}^2 = x^2 + y^2 + z^2$
- Here $\|\vec{x}\|$ is the Euclidean length
- Norm (squared) of a four vector:

$$\|x_\mu\|^2 = x_0^2 - (x_1^2 + x_2^2 + x_3^2)$$
$$= (ct)^2 - (x^2 + y^2 + z^2) = (ct)^2 - \vec{x}^2$$

- The norm squared is a conserved quantity in Lorentz transformations.

Space and Time

## Three-Momentum and Energy

- In Newtonian physics the energy $E$ and three-momentum $\vec{p}$ are conserved quantities
- In SR extended to four-momentum $p^\mu$ conservation in each component:
  - $E$ with the addition of rest mass energy
  - three components of $\vec{p}$
- In a closed system the sum of all particle does not change, can be used to calculate the kinematics of some processes, where a set of incoming particles produces a set of outgoing particles

$$\sum_{\text{in}} p^\mu_{\text{in}} = \sum_{\text{out}} p^\mu_{\text{out}}$$

- Squares of four-vectors are generally interesting, as they are invariant under Lorentz transformation
- Calculate square of $p^\mu$:

$$p^\mu = (E/c, \vec{p})$$
$$\|p^\mu\|^2 = (p^\mu)^2 = p^\mu p_\mu = E^2/c^2 - \vec{p}^2$$

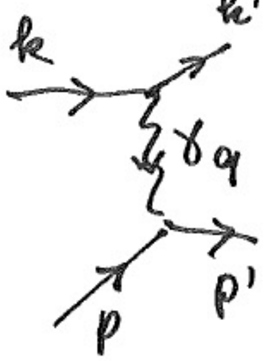
- In the rest frame of an object:
  - $\vec{p} = 0$
  - $E = mc^2$
- Thus:

$$p^\mu p_\mu = m^2c^2 = E^2/c^2 - \vec{p}^2$$
$$E = \sqrt{m^2c^4 + \vec{p}^2c^2}$$

- Square of the four momentum is equal to the invariant (rest) mass of a system$\cdot c^2$!

# 4-momenta of scattering diagram kinematics

① $|T_\gamma|^2$




Neglect masses

$$(k+p)^2 = 2kp = (k'+p')^2 = 2k'p' = s$$

$$s^2 = 4(kp)(k'p')$$

$$(k-p')^2 = (k'-p)^2 = -2kp' = -2k'p = u$$

$$u^2 = 4(k'p)(kp')$$

$$t = (k-k')^2 = q^2 = -Q^2$$

$$s+t+u=0$$

$$s^2+u^2 = 2s^2+t^2+2st \quad , \quad s^2 u^2 = -t^2-2st$$

# Amplitudes

$$T_\gamma = \frac{1}{(2\pi)^{3/2}} \bar{\mu}^{s'}(k') \, ie \, (2\pi)^4 \delta(k-k'-q) \gamma^m \mu^s(k) \frac{1}{(2\pi)^{3/2}}$$

$$\cdot \, i \int \frac{dq}{(2\pi)^4} \frac{1}{q^2+i\varepsilon} g^{mn}$$

$$\frac{1}{(2\pi)^{3/2}} \bar{q}^{r'}(p') \, ie \, (2\pi)^4 \delta(p'-p-q) \gamma^n Q_q \, q^r(p) \frac{1}{(2\pi)^{3/2}}$$

---

$$T_\gamma = \frac{-ie^2 Q_q}{(2\pi)^2 q^2} \bar{\mu}^{s'}(k') \gamma^m \mu^s(k) \cdot \bar{q}^{r'}(p') \gamma^m q^r(p)$$

$$T_\gamma^+ = \frac{ie^2 Q_q}{(2\pi)^2 q^2} \bar{q}^r(p) \gamma^m q^{r'}(p') \; \bar{\mu}^s(k) \gamma^m \mu^{s'}(k')$$

---

Feynman Rules: Bogoljubov,Shirkov: Quantum Field Theory

# Amplitude Product

$$\frac{1}{2}\sum_{s}\;\frac{1}{2}\sum_{r} \qquad \sum_{s',r'}$$

average

Average over incoming spins
Sum over outgoing spins

$$T^2 = T_\gamma T_\gamma^+ + h.c.$$

$$\sum_{s'} \mu^{s'}(k')\,\bar{\mu}^{s'}(k') = k'^l \gamma^l = \hat{k}' \quad , \quad \int dk' = \int \frac{d^3k'}{2k_0'}$$

$$\overline{T_\gamma^+ T_\gamma} = \frac{1}{4}\,\frac{e^4 Q_q^2}{(2\pi)^4 t^2}\left\{\sum_s \bar{\mu}^s(k)\,\gamma^n\,\hat{k}'\,\gamma^m\,\mu^s(k)\right\}$$

$$\cdot\left\{\sum_r \bar{q}^r(p)\,\gamma^n\,\hat{p}'\,\gamma^m\,q^r(p)\right\}$$

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Trace of γ matrices

$$\left\{\sum \quad\right\} = tr\left(\gamma^n k'^l \gamma^l \gamma^m k^p \gamma^p\right)$$

$$= 4\left(k'^n k^m + k^n k'^m - g^{nm} k k'\right)$$

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$$\cdot\; 4\left(p'^n p^m + p^n p'^m - g^{mn} p p'\right)$$

# 4-Vector Algebra

$$= \underline{4\left(k'^n k^m + k^n k'^m - g^{nm} kk'\right)}$$

$$\cdot \; 4\left(p'^n p^m + p^n p'^m - g^{mn} pp'\right)$$

$$= 16\left((k'p')(kp) + (kp')(k'p) + \cancel{kp} - (p'p)(kk') \right.$$

$$+ (k'p)(kp') + (kp)(k'p') - (pp')(kk')$$

$$\left. - (k'k)(pp') - (kk')(pp') + 4(kk')(pp')\right)$$

$$= \underline{32\left[(k'p')(kp) + (k'p)(kp')\right]}$$

# Product of Amplitudes

$$\overline{T_\gamma^+ T_\gamma} = \frac{1}{4} \frac{e^4 Q_q^2}{(2\pi)^4 t^2} \left\{ \sum_s \bar{\mu}^s(k) \gamma^n \hat{k}' \gamma^m \mu^s(k) \right\}$$

$$\cdot \left\{ \sum_r \bar{q}^r(p) \gamma^n \hat{p}' \gamma^m q^r(p) \right\}$$

$$= 32 \left[ (k'p')(kp) + (k'p)(kp') \right]$$

$$= 8 \cdot 4 \left[ \quad + \quad \right]$$

$$\overline{T_\gamma^+ T_\gamma} = \frac{1}{4} \frac{e^4 Q_q^2}{(2\pi)^4 t^2} 8 \left( s^2 + u^2 \right)$$

cf above

# Cross Section Kajantie Byckling

<u>total cross section</u> $p_a + p_b \to p_1 \cdots p_n$ , $(p_a+p_b)^2 = s,\ p_i^2 = m_i^2$

$$\sigma_{tot}^{(n)}(s, m_i) = \frac{1}{\phi} \cdot \underbrace{\int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i}\, \delta^4(p_a + p_b - \Sigma p_i)}_{\text{phase space integral } (T=1)} T(\vec{p}_i) = \frac{I_n}{\phi}.$$

$\phi$ : "Flux factor" $= \phi = 2\lambda^{1/2}(s, m_a^2, m_b^2)\underbrace{(2\pi)^{3n-4}}$ ← often included in $I_n$.

$$(\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx).$$

<u>differential cross section</u> $[a + b \to 1 \ldots n]$

$$\frac{d\sigma_n}{dx} = \frac{1}{\phi} \int \prod_{1}^{n} \frac{d^3 p_i}{2E_i}\, \delta^4(p_a + p_b - \Sigma p_i)\, \delta(x - x(\vec{p}_i)) \cdot T(\vec{p}_i).$$

# Elastic lp Scattering Cross Section

$$\frac{d\sigma}{dt} = \frac{2}{\sqrt{\Delta}} (2\pi)^{3\cdot 2-4} \int \tilde{d}p\, \tilde{d}p'\, \delta^4(k-k'+p-p')\, \delta(t-(k-k')^2) |T_\gamma|^2$$

$$= \frac{2}{\sqrt{\Delta}} (2\pi)^2 \frac{\pi}{8\sqrt{\Delta}} \cdot |T_\gamma|^2 = \frac{\pi^3}{s^2} |T_\gamma|^2 = \frac{d\sigma}{dt}$$

$\Delta = s^2$

$$\frac{d\sigma}{dt} = \frac{2\pi e^4 Q_q^2}{16\pi^2 t^2} \frac{s^2+u^2}{s^2} \qquad \alpha = \frac{e^2}{4\pi}$$

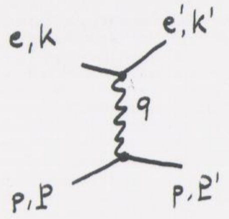
$$\boxed{\frac{d\sigma}{dt} = \frac{2\pi\alpha}{t^2} \cdot Q_q^2 \cdot \frac{s^2+u^2}{s^2}}$$

$\mu q \to \mu q$

unpolarised.

29.10.79

## Elastic ep Scattering Cross Section




$s = (k+P)^2 \qquad t = q^2 = (k-k')^2 = -Q^2$

Mandelstam: $s + t + u = 0$, $u = (k-P')^2$

scattering amplitude: $T = -i \frac{e^2}{(2\pi)^2} \cdot j^m_{elm}(e) \cdot \frac{1}{q^2} \cdot j^m_{elm}(p)$
pointlike e and p

$j^m_{elm}(e) = \bar{e}^{s'}(k') \cdot \gamma^m e^s(k) \qquad j^m_{elm}(P) = \bar{p}^{r'}(P') \gamma^m p^r(P)$

average over incoming spins s, r. sum over s', r'

$\sum_{s'} e^{s'}(k') \bar{e}^{s'}(k') = k'^n \gamma^n := \hat{k'}$

cross section $\frac{d\sigma}{dt} = TT^+ \cdot \frac{\pi^3}{s^2}$ amplitude$^2$ · phase space

use $\sum_s \bar{e}^s(k) \gamma^m \hat{k'} \gamma^m e^s(k) = tr(\gamma^n k'^\ell \gamma^\ell \gamma^m k^p \gamma^p) = 4(k'^n k^m + k^n k'^m - g^{mn} kk')$
elm. tensor

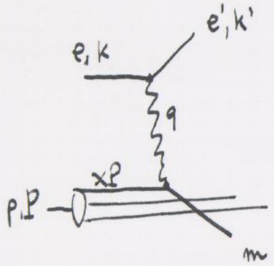
$\frac{d\sigma}{dt\,du} = \frac{2\pi\alpha^2}{t^2} \cdot \underbrace{\frac{s^2+u^2}{s^2}}_{\approx 2 \text{ for } E \ll M} \cdot \delta(s+t+u) \qquad \alpha = e^2/4\pi$

elastic $eP \to eP$ cross section exactly calculable in QED.

depends only on $s = 2ME_e$ and $t = -Q^2 = -4E^2 \sin^2\frac{\theta}{2}$

or the incoming energy $E_e$ and the scattering angle of e, $\theta$

## Inelastic ep Scattering in the QPM




$(xP+q)^2 = m^2 \quad . \quad P^2 = M_p^2$

if $q^2 \gg x^2 M^2, m^2$ : $q^2 + 2xPq = 0$

$x = \frac{Q^2}{2Pq}$ fraction of p momentum carried by quark

fixed target: $P = (M_p, \vec{0})$: $2Pq = 2M_p \cdot \nu = 2M_p \cdot E_e \frac{\nu}{E_e} = sy$

$\nu = E_e - E_e'$ energy transfer

inelastic ep cross section: $p = xP$ : $s = xS \quad t = T \quad u = xU$

$\delta(u+t+s) \to \sum_{\text{quarks}} \times \underbrace{\delta\left(x + \frac{T}{S+U}\right)}_{q(x)+\bar{q}(x)} \cdot e_q^2 \quad := F_2(x)$

$q(x)+\bar{q}(x)$: momentum distribution of quarks and antiquarks in p.

$e_q$: charge of quark in units of e

$\frac{S^2+U^2}{S(S+U)} = \frac{Y_+}{y} \qquad Y_+ = 1 + (1-y)^2$

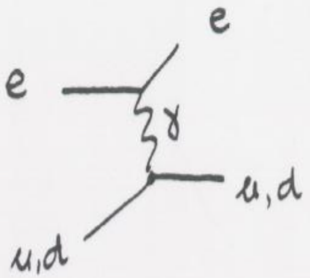
$\frac{d\sigma}{dQ^2\,dx} = \frac{2\pi\alpha^2}{Q^4 x} \cdot Y_+ F_2(x)$

Bjorken scaling

$F_2(Q^2, \nu) \to F_2\left(x = \frac{Q^2}{2M\nu}\right)$

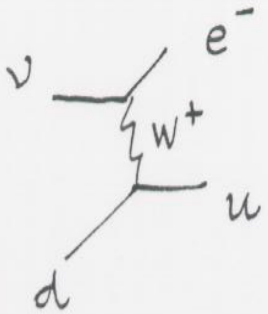
$Q^2, \nu$ large.

# Verification of the Quark-Parton Model - Fractional Electric Charges




neutral current

(elm: $\gamma$)




charged current

(weak!)

HERA:


inverse charged current.

$$F_2 = x \sum e_q^2 (q + \bar{q})$$

proton: uud

neutron: ddu

$e_u = 2/3 \quad e_d = -1/3$

$$F_2^{\nu p} = 2x \left[ d + \bar{u} \right]$$

$$F_2^{\nu n} = 2x \left[ u + \bar{d} \right] \qquad F_2^{\nu N} = x \left[ u + \bar{u} + d + \bar{d} \right]$$

$$F_2^{p} = x \left[ e_u^2 (u + \bar{u}) + e_d^2 (d + \bar{d}) \right]$$

$$F_2^{n} = x \left[ e_d^2 (u + \bar{u}) + e_u^2 (d + \bar{d}) \right]$$

$$\rightarrow \frac{F_2^{eN}}{F_2^{\nu N}} = \frac{1}{2} \left( e_u^2 + e_d^2 \right) = \frac{5}{18} \simeq 0.28$$

$$\frac{1}{2} \left( F_2^{ep} + F_2^{en} \right) = F_2^{eN}$$

$$= x \, \frac{e_u^2 + e_d^2}{2} \left[ u + \bar{u} + d + \bar{d} \right]$$

$$\frac{\text{GGM } \nu N}{\text{SLAC } eN} : \; 0.29 \pm 0.05$$

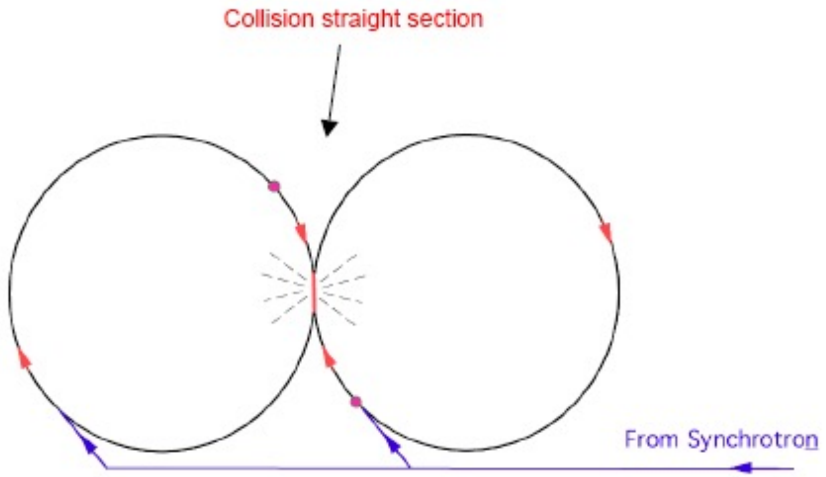
# Storage Rings

**D.W. Kerst et al** *"The possibility of producing interactions in stationary coordinates by directing beams against each other has often been considered, but the intensities of beams so far available have made the idea impractical.*

*............... accelerators offer the possibility of obtaining sufficiently intense beams so that it may* ***now*** *be reasonable to* ***reconsider directing two beams*** *of approximately equal energy* ***at each other.****"*

*D. W. Kerst et al., Phys. Rev. 102, 590 (1956).*

**G. K. O'Neill**, interested in p-p collisions, introduces the idea of injecting the beam extracted from a high energy proton synchrotron in two "**storage rings**" in which particles would be accumulated and stored for a long time. Typically in a figure-of-8 configuration they have a common section in which the two stored beams collide head-on.




Tazzari, Cern Acc School 2006

fixed target accelerator: $s=2ME$, collider: $s=4E^2$ : gain: $2E/M$

First $e^+e^-$ storage ring ADA at Frascati: Bruno Touschek et al.