# The Physics of Damping Rings, Part 1

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**Abstract**

This is a review on the basics physics of damping rings, expressed in a way that is simple and intuitive. My approach is to bring together just enough of the basic ingredients in accelerator physics to calculate and observe some of the common features like closed orbit, dispersion and frequency map.

## 1 Introduction

This report contains the notes on the basics of damping rings that I have been writing on and off over a period of a few months for my web page. [1]

My approach to understanding the physics of damping rings is to bring together just enough of the basic ingredients in the accelerator physics of the damping ring, to calculate and observe some of the common features. I intend to do this using just Newtonian mechanics, which is more intuitive to me. No Hamiltonian mechanics is used.

The ingredients would include bending by magnetic fields, energy loss due to synchrotron radiation, and energy gain in the rf cavity. These are used to derive the matrix elements for dipoles, quadrupoles, sextupoles, rf cavities, etc. The matrices help us to calculate the change in position and momentum of an electron (or positron) each time it passes through a magnet, a cavity or a drift space on the ring. I shall use the thin lens approximation for all of these and see how far I can get. Using these, I hope to be able to calculate features like betatron oscillations, closed orbit, dispersion, radiation damping, synchrotron oscillations, phase stability, and maybe even a frequency map.

[1] http://hep.ph.liv.ac.uk/∼hock/Damping_Ring/Damping_Ring.html

Instead of just taking notes from papers and books, I would like to turn this effort into an exercise with an objective. The objective of this exercise is twofold: to work out what is the smallest set of elements required to produce the above features, and to understand the basic physics of each matrix from first principles. I am guided in this undertaking by the endless patience of Dr Desmond Barber [1], and a paper by Alex Chao [2].

This is a work in progress. I intend to include the following items:

(i) Synchrotron Radiation
(ii) Radiation Damping
(iii) Bending magnet
(iv) Kicker magnet
(v) rf cavity
(vi) Drift Space
(vii) Quadrupole magnet
(viii) Sextupole
(ix) 7 by 7 matrix method
(x) Closed orbit
(xi) Dispersion
(xii) Frequency Map

The present article is essentially part 1, and consists of items (i) to (v).

# 2 Synchrotron Radiation

The damping in a damping ring is caused by radiation. If this seems difficult to understand, it is precisely the reason why I choose to start with this topic.

When an electron goes round the ring, it accelerates towards the centre of the ring. The acceleration causes it to emit radiation. Whenever an electron emits radation, it recoils, just like a gun would recoil when a bullet is fired. Broadly speaking, this recoil effectively reduces the energy of the electron. As a result, the oscillation about its reference path is reduced, so that we get a narrower beam. This is what damping means.

In subsequent pages, we shall see that things are a little more complicated than that, because the rf cavities in the ring would put energy back into the electron, causing its momentum to increase in the forward direction. For now, I shall focus on how to calculate the energy loss, commonly known as radiation loss.

Radiation is emitted when the electron passes through a bending magnet. In this section, I shall go through the steps to calculate the resulting change in energy. The steps are based on references [3], [4], [5], [6], [7], [8], [9].

The notes on this page is motivated by the need to calculate the radiation loss when an electron passes through a bending magnet. This is needed for the matrix element of the magnet, which would be covered in section 4. To prepare for this, I derive here the formula for the rate of radiation loss for an electron when it revolves around a circle. This is the formula:

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 a^2 \tag{1}$$

where

$P$ is the rate of radiation loss,

$q$ is the electron charge,

$a$ is the acceleration, and

$\gamma$ is the energy, given by $1/\sqrt{1 - v^2/c^2}$.

The derivation of this equation may be divided into two parts. Part 1 derives the formula for Larmor radiation, which gives the power radiated from an accelerating point charge in the non- relativistic limit. Part 2 finds the Lienard's formula, the relativistic form of the Larmor radiation formula. This would be the above formula, which can then be used when an electron is close to light speed.

## Part 1: Larmor Radiation

I have initially followed Jackson's derivation [8] of the Larmor radiation formula from Maxwell's equations. It is very rigorous and mathematical.

I have subsequently found another derivation which is much simpler. Less rigorous perhaps, but very intuitive. The derivation is attributed to J. J. Thomson. I shall go through it here.

I found this derivation on a website on radio astronomy. It is interesting to learn that the same principles are used in very different fields.

Figure 1 illustrates the electric fields around an electron that is initially at rest. It is accelerated for a short time $\Delta t$ to a velocity $\Delta v$. It carries on at that same velocity after that. The result of this sudden "jerk" is that a kink is created in the electric field around the electron. This kink travels outward at the speed of light, and form the radiation that is generated by the short acceleration. We need to derive the radiation flux in the kink.

There is some interesting physics about this kink that should be clarified before we proceed:

(i) When the electron travels at a constant velocity, the electric field lines travel with it.

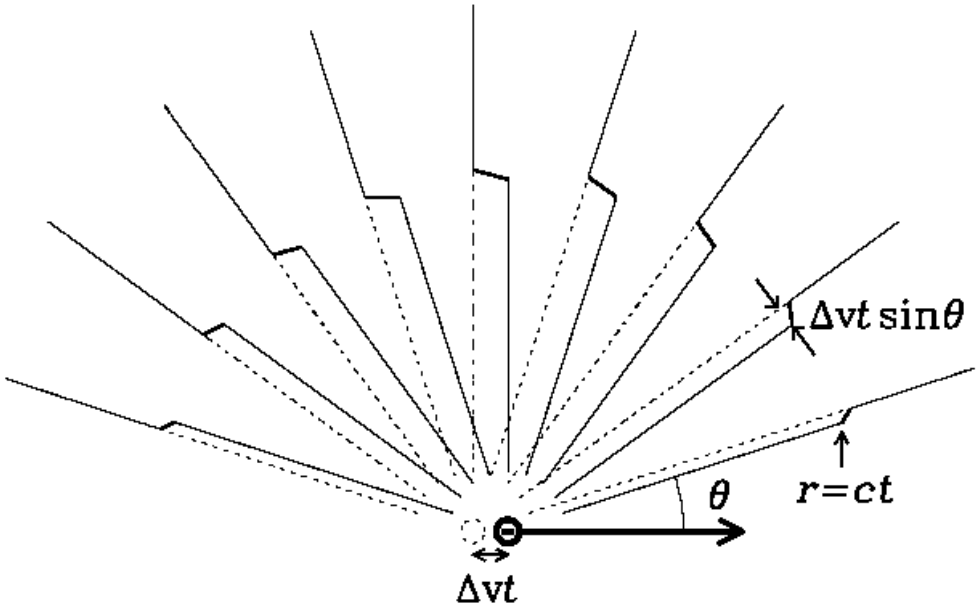


Figure 1: When an electron is accelerated, it carries its electric field along. The field further would only experience this drag at a slightly later time because signal cannot travel faster than light.

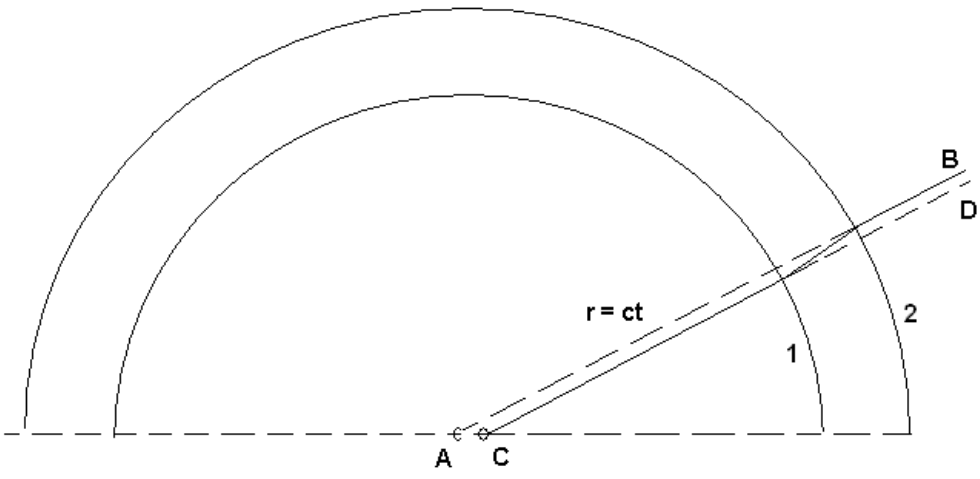


Figure 2: The effect of accelerating the electron for a short time is to produce a kink in the field that travels outward at the speed of light.

(ii) Every electric field line (e.g. CD in fig. 2) remains parallel to its original direction (e.g. AB).

(iii) Outside a sphere of radius $ct$, the electric field lines are from the electron at rest.

(iv) Inside a sphere of radius $c(t - \Delta t)$, the electric field lines are from the electron at constant velocity.

(v) The layer in between the two spheres is the kink.

(vi) The two spheres, and therefore the kink, travels outward at the speed of light.

(vii) The two spheres are approximately concentric because the electron is so much slower than light.

(viii) For the same reason, the length of the kink is much greater than the distance AC traveled by the electron.

(ix) The length of the kink, however, is much smaller than the radius of both spheres, since $\Delta t$ would be much smaller than $t$ after some time.

(x) Therefore the radius of the two spheres would be approximately equal, to $ct$.

(xi) This kink has transverse electric field component - transverse as in perpendicular to the original field line.

(xii) The appearance of this component in turn induces magnetic field.

(xiii) Together, they form the electromagnetic wave, or radiation.

We need to estimate this transverse electric field component. In order to do so, consider the kink at a time $t$. The kink has been exaggerated in the figure above. Since the kink is really just a slight distortion of the electric field, it is reasonable to expect that the magnitude of the electric field along the kink remain approximately the same as the Coulomb field of the electron at rest: $q/r^2$. For the same reason, the radial component of the field is also approximately the same, i.e.

$$E_r = \frac{q}{r^2} \tag{2}$$

With the help of the triangle FGH enclosing the kink in fig. 3, and using similar triangles, we can see that

$$\frac{E_\perp}{E_r} = \frac{q}{r^2} \tag{3}$$

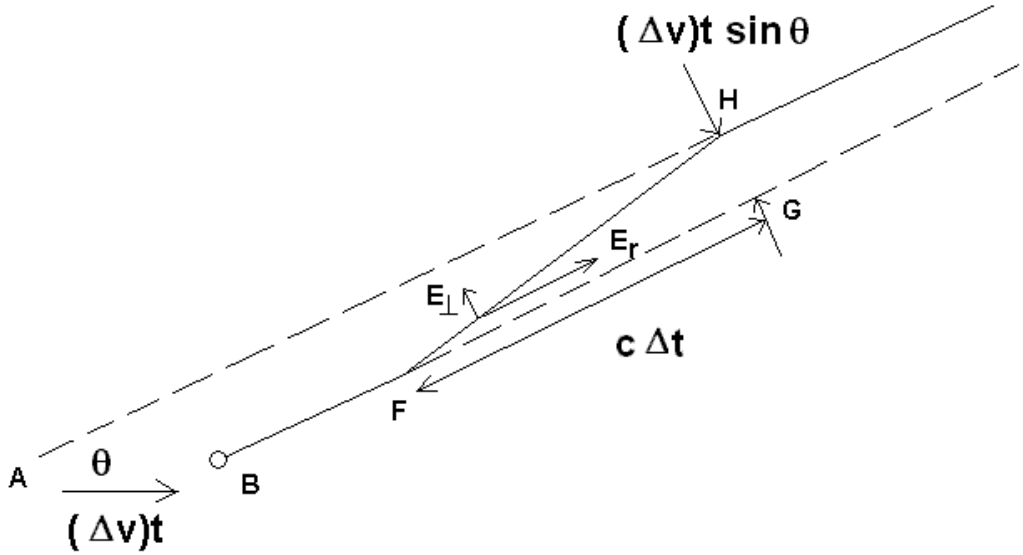


Figure 3: A magnified view of the kink.

Note that the distance AB is approximated as $(\Delta v)t$ because the distance travelled during the initial $\Delta t$ has been neglected. Substituting $t = r/c$, the transverse component can now be obtained:

$$E_\perp = \frac{qa \sin\theta}{rc^2} \tag{4}$$

where the acceleration $a = \Delta v/\Delta t$. Using the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \tag{5}$$

and taking $|\mathbf{E}| = |\mathbf{B}|$ (cgs units for plane wave, true if the particle is oscillating sinusoidally), so that

$$S = \frac{c}{4\pi}E^2 \tag{6}$$

the energy flux in direction $\theta$ can be derived:

$$S = \frac{1}{4\pi}\frac{q^2a^2}{c^3}\frac{\sin^2\theta}{r^2} \tag{7}$$

Note that this result predicts that radiation is maximum in a direction perpendicular to the acceleration of the electron. The total power can now be obtained by integrating over all direction. This gives:

$$P = \frac{2}{3}\frac{q^2a^2}{c^3} \tag{8}$$

This is Larmor's equation.

### Part 2: Lienard's Formula

Here, I shall go through a simple derivation. Again, a rigorous treatment is available in Jackson (1999). The result is this: when an electron goes round a circle at relativistic speed, the acceleration in Larmor's equation is increased by a factor of $\gamma^2$.

Suppose that in the lab frame, the electron is is moving in the x direction at a particular instance in time, and accelerating in the y direction. In the rest frame, the electron velocity would be zero, but it would still be accelerating in the y direction. This is where Larmor's equation is valid.

Then we do a Lorentz transformation to the lab frame. The simplest way to understand this is from the definition of acceleration: $a = d^2x/dt^2$. Because of time dilation, the $dt^2$ in the denominator gives the factor of $\gamma^2$. That's it.

The reason why it is that simple is because when the electron acceleration is purely perpendicular to the relative motion along x. If you are not convinced, you can look at the full Lorentz transformation equations for acceleration. It is not easy to find this, as textbooks often gives only the transformation for the x component. However, you can find the complete set in [7]. Just set all the x components to zero, and you would get the same result.

Once we are convinced that the acceleration, $a$, gains a factor of the $\gamma^2$ when we transform to the lab frame, we can insert these into Larmor's equation and get

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 a^2 \tag{9}$$

This is Lienard's formula for the power from synchrotron radiation.

This formula can now be used to derive the matrix element needed to calculate radiation damping in the damping ring.

# 3 Radiation Damping

Radiation damping is usually explained as a combination of two effects: energy loss when an electron goes pass a bending magnet, and energy gain when it goes through an rf cavity. In this section, I shall derive the energy loss that results when it goes pass a bending magnet. This will be used in the transfer map for a bending magnet. The cause of this energy loss is the emission of radiation as a result of the acceleration in the magnet.

This is the formula that I want to derive:

$$\delta = -C_\gamma E_0^3 \left(\frac{l}{B\rho} B_y\right)^2 (2\pi l)^{-1} \tag{10}$$

where

$\delta$ is the fractional change in energy,

$E_0$ is the energy of the electron,

$l$ is the length of the path in the magnetic field,

$B_y$ is the magnetic field (assumed uniform),and

$B\rho$ is the beam rigidity.

For our purpose, we consider the simplest example in which $B$ is the same as $B_y$, and $\rho$ is the radius of the circular arc traced out by the electron in the magnetic field. $C_\gamma$ is a constant that is commonly used in formulae for calculating synchrotron radiation. It is defined as:

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} \tag{11}$$

where $r_e = e^2/m_e c^2$ is the classical electron radius. The numerical value of $C_\gamma$ is $8.85 \times 10^{-5}$ meter-GeV$^{-3}$. This definition is given in [9].

We start with Lienard's formula for synchrotron radiation power:

$$P = \frac{2}{3} \frac{q^2}{c^3} \gamma^4 a^2 \tag{12}$$

First, we convert to Sands's version [9], which is often used in accelerator physics calculations:

$$P_\gamma = \frac{cC_\gamma}{2\pi} \frac{E^4}{\rho^2} \tag{13}$$

where $P_\gamma$ is the same as $P$.

To arrive at this, we have to make use of $E = \gamma m_e c^2$ and the centripetal acceleration formula $a = v^2/\rho$, where $v = c$ for the electron moving close to light speed.

Next, to get the energy radiated, we multiply $P_\gamma$ by the time taken to traverse the magnet, which is $t = l/c$. The energy radiated is equal to the change in energy of the electron, which is then given by

$$\Delta E = P_\gamma t = \frac{cC_\gamma}{2\pi} \frac{E_0^4}{\rho^2} \frac{l}{c} \tag{14}$$

We now use $E_0$ for the energy in place of E, to migrate over to the notation for transfer maps in Chao (1979). After some rearranging, we arrive at:

$$\frac{\Delta E}{E_0} = C_\gamma E_0^3 \Big( \frac{l}{B\rho} B \Big)^2 (2\pi l)^{-1} \tag{15}$$

Since the right had side is the fractional energy change $\delta$ and since B is taken to be the same as $B_y$, this is the same equation as the Eq. 10 that I want to derive.

This can now be used to obtain the transfer map for the bending magnet.

# 4 Horizontal Bending Magnet

The purpose of bending magnets are to change the direction of the beam. They are placed at a number of locations on the ring to guide the beam along the reference path. Ideally, each bending magnet would produce a uniform, vertical magnetic field through the beam pipe. This field is also called the dipole field.

The objective of this page is to understand the matrix elements for the transfer map of this dipole element, using - yes - Newtonian mechanics. The matrix elements for the horizontal bending magnet in [2] are listed here:

$$m_{26} = -m_{51} = \Big(\frac{l}{B\rho}\Big)B_y \tag{16}$$

$$m_{67} = -C_\gamma E_0^3\Big[\Big(\frac{l}{B\rho}\Big)B_y\Big]^2\Big(\frac{1}{2\pi l}\Big) \tag{17}$$

$$m_{27} = \frac{1}{2}\Big[\Big(\frac{l}{B\rho}\Big)B_y\Big]m_{67} \tag{18}$$

First, I shall briefly explain the what the transfer map - or matrix - does. It is supposed to give us the new coordinates of a particle just after it goes past the element - in this case the magnet. The coordinates just before the element must be given. Note that the origin of these coordinates is the ideal position of the particle on the reference path. So as time passes, this origin actually moves along the path, and the coordinates are in fact the deviation from this path. The coordinates are arranged to form a vector:

$$\mathbf{x} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \\ 1 \end{pmatrix} \tag{19}$$

The meaning of each coordinate is as follows:

$x$ is the horizontal displacement

$x'$ is the gradient of x with respect to z

$y$ is the vertical displacement

$y'$ is the gradient of x with respect to z

$z$ is the displacement along the reference path

$\delta$ is the fractional energy deviation

What about the 1 at the bottom? Well, it is there to make it easier to write the matrix equation. This should be clear by the end of the page.

Suppose that this vector is $\mathbf{x}_0$ just before the magnet, and $\mathbf{x}_1$ just after the magnet. Suppose that $\mathbf{M}$ is the transfer matrix. Then these three quantities are related by:

$$\mathbf{x}_1 = \mathbf{M}\mathbf{x}_0 \tag{20}$$

The transfer matrix $\mathbf{M}$ looks like this:

(i) The diagonal elements are all equal to 1.

(ii) The other elements are all zeroes, except

(iii) those listed in Eqs. 16 to 18 above.

Just to make it very clear, this is what it really looks like:

$$\begin{pmatrix} x_1 \\ x'_1 \\ y_1 \\ y'_1 \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{21}$$

The first and possibly the most important thing to understand about this transfer matrix is this. Suppose $\mathbf{x}_0$ is zero. That is, the particle is following the ideal path perfectly just before it enters the magnet. According to Eq. 20, $\mathbf{x}_1$ is also zero.

Now if, as I have explained, the magnet changes the direction of the particle, why are the coordinates in $\mathbf{x}_1$ not changed at all?

The reason is: the reference path just after the magnet is also bent by the same amount. The magnet is designed to guide the particle along the reference path. If the particle is on the reference path before the magnet, the

magnet is designed to bend it by just the right amount so that it continues along the reference path when it leaves.

So to understand **M**, it is important to bear in mind that it is supposed to give the deviations of the coordinates from this bent reference path, and not the deviation from a straight line. So $\mathbf{x}_1 = 0$ means that the electron is still following the design path.

Equation 21 really consists of six equations as follows:

$$\begin{aligned} x_1 &= x_0 && (22)\\ x_1' &= x_0' + m_{26}\delta_0 + m_{27} && (23)\\ y_1 &= y_0 && (24)\\ y_1' &= y_0' && (25)\\ z_1 &= z_0 + m_{51}x_0 && (26)\\ \delta_1 &= \delta_0 + m_{67} && (27) \end{aligned}$$

Equations 24 and 25 are the easiest to understand. Because the magnet is horizontal bending, it exerts no force in the vertical direction. So neither the y coordinate nor gradient (y') are changed by the magnet.

Equation 22 involves an approximation. The magnet causes the particle to bend in its path. However, because the particle is travelling close to light speed, the time spent within the field is very short. So the actual displacement in x may be neglected. It can be proven that this is true to first order, but we shall not go into that here.

Let us start with Eq. 27. This tell us the new fractional energy deviation, after going through the magnet. Why would the energy change when the electron goes through the magnetic field? Because it is accelerated sideways by the field, causing it to emit radiation. The radiation energy that is emitted has been derived in section 2. It is given by:

$$\frac{\Delta E}{E_0} = -C_\gamma E_0^3 \Big(\frac{l}{B\rho}B\Big)^2 (2\pi l)^{-1} \qquad (28)$$

The minus sign is there because the electron loses energy. This is precisely the same expression as that in Eq. 17. Thus, the matrix element $m_{67}$ is in fact the fractional energy change due to radiation. By adding this change to the the factional energy before the magnet, we obtain Eq. 27.

I shall now look at Eq. 23. This involves the element $m_{26}$ from Eq. 16 above. This element relates the energy change just before the magnet, to the horizontal gradient just after the magnet. Figure 4 will help us see how this is possible.

The path AB represents the path of the particle in the magnetic field. In a uniform field, this path is a circular arc. Let O be the centre of this arc.

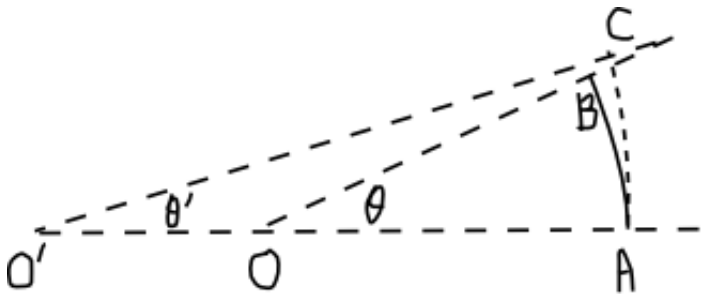


Figure 4: The arcs AB and CD covers the extent of the magnetic field. O and O' are the respective centres of the arcs.

The radius, r, is related to the energy and field by the following centripetal force equation:

$$\frac{\gamma m_e v^2}{r} = B_y e v \tag{29}$$

So if the energy, $\gamma$, increases, the radius increases. As a result, the path becomes straighter. In the figure above, the new path is illustrated by AC, which has a new centre O'. As it turns out, the original angle, $\theta$, is actually equal to the matrix element $m_{26}$.

To see why this is so, look at Eq. 16 again. In the magnetic field, B is $B_y$ and $\rho$ is r. Therefore,

$$m_{26} = \frac{l}{r} = \theta \tag{30}$$

We have taken $r$ to be OA and $l$ to be the path length AB. According to Eq. 29, if $\gamma$ increases by a fraction $\delta_0$, so does $r$. The new radius O'A is therefore equal to $r(1 + \delta_0)$.

The new arc length AC is about the same as the old arc AB, since they are determined by the extent of the same field. To first order, for the same arc length, an increase in radius by a fraction of $\delta$ gives a decrease in angle by the same fraction. So $\theta'$ is smaller than $\theta$ by $\theta\delta_0$. I shall redraw fig. 4 in a very simple way in 5.

I have used right angled triangles to approximate the two sectors. One can easily think of more accurate ways of doing this, but it would lead to the same answer, so I shall sacrifice rigour for clarity.

It is immediately obvious from this figure that the original path AB, has changed direction by an angle of $\theta\delta_0$, to AC. This is also the change in gradient of the particle, so that

$$x_1' = x_0' + \theta\delta_0 \tag{31}$$

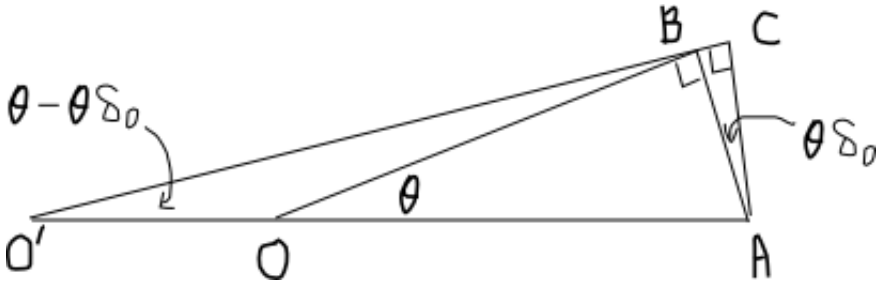


Figure 5: A simplified diagram with arcs AB and AC approximated by straight lines.

Combining with Eq. 30, we get

$$x_1' = x_0' + m_{26}\delta_0 \tag{32}$$

This is not quite Eq. 23 yet. There is still the $m_{27}$ term. Lets look at Eqs. 18 and 27for some clue. The element $m_{27}$ contains $m_{67}$. From the previous paragraphs, we know that this is the energy change due to radiation. It also contains the expression for $m_{26}$ - by comparing with Eq. 16. From the Eq. 32, we know that when multiplied by energy change, this gives us the deviation in $x'$. Lets write down what we have learnt about Eq. 18 so far:

$$m_{27} = \frac{1}{2} m_{26} m_{67} \tag{33}$$

Then there is the factor of 1/2. The radiation takes place continuously (in the classical approximation, if we don't worry about photons for now) as the electron moves through the magnetic field. It is reasonable to expect that the average energy change due to radiation is about half of the final value of $m_{67}$.

So, to include the effect of this radiation, lets add half of $m_{67}$ to the energy change $\delta_0$ in Eq. 32:

$$\begin{aligned} x_1' &= x_0' + m_{26}\left(\delta_0 + \frac{1}{2} m_{67}\right) \\ &= x_0' + m_{26}\delta_0 + \frac{1}{2} m_{26} m_{67} \\ &= x_0' + m_{26}\delta_0 + m_{27} \end{aligned}$$

where I have made use of Eq. 33. This finally gives us Eq. 23.

Lets look at Eq. 26 now. This is saying that if there is a deviation in $x$ just before the magnet, $z$ will be different when the electron leaves the magnet. How is this possible?

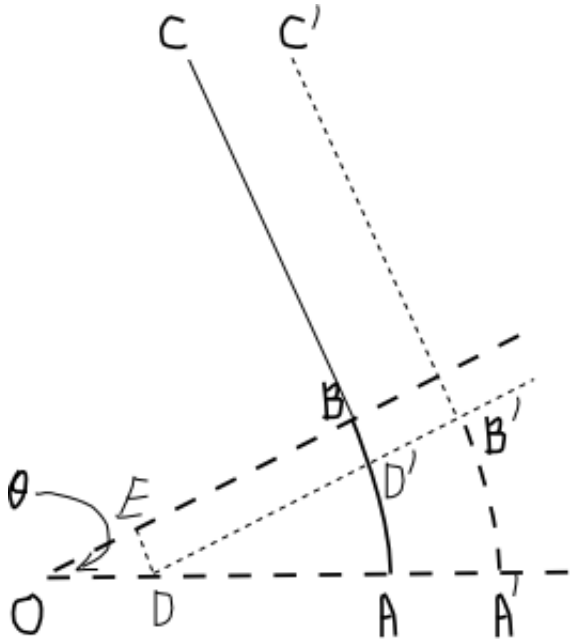


Figure 6: In the magnetic field, an electron closer to the centre of theh design path comes out ahead of one that is further.

In fig. 6, ABC is the reference path that the electron would follow if there is no deviation in $x$ before the magnet. The boundaries of the field is defined by the dashed lines along OB and OA.

If there is no deviation, the electron enters the field at A. If there is deviation, suppose that the electron enters at A'. When the electron at A reaches B, the electron at A' would reach B'. The electron at B would then continue along BC, while the electron at B' would continue along B'C'

With respect to the electron at B, the electron at B' obviously lags behind - by the amount D'B. This is the deviation in $z$.

We can obtain an expression for this deviation by looking at the triangle ODE. DE is equal to D'B, the z deviation. Since $\theta$ is generally small, DE would be close to OD times $\theta$. Since OD is equal to AA', which is the deviation $x_0$, we have

$$\begin{aligned} z_1 &= z_0 - x_0\theta \\ &= z_0 - x_0 m_{26} \\ &= z_0 + x_0 m_{51} \end{aligned}$$

where I have made used of Eqs. 30 and 16. This gives us Eq. 26.

This concludes the section on horizontal bending magnet. Vertical bending magnet follows very similar reasonings.

# 5 Horizontnal Kicker Magnet

The purpose of the kicker magnet is to change the direction of the beam. But then I said the same thing about the bending magnet. So what is the difference?

Physically, there is no difference, except possibly for the strength. For calculation purpose, however, there is.

It has to do with the way that the calculation is usually carried out. Usually, we define a reference path, and calculate the actual trajectory in terms of the deviation from this path.

For a bending magnet, if a particle enters along the reference path (and that also means zero z and $\delta$ deviations), it will leave along the reference path. This is because the bending magnet is designed to bend the trajectory along the reference path.

For a kicker magnet, however, if a particle enters along the reference path, it will not leave along the reference path. This is because the kicker magnet is designed to bend the trajectory away from the reference path - fig. 7.

But why do it differently from a bending magnet? Because the strengths of kicker magnets are adjusted to compensate for misalignments of of other elements, such as quadrupole magnets. Initially, the strengths are not known, so we can only assume a straight reference path through each kicker magnet.

Most of the matrix elements for the kicker are the same as those of the bending magnet. There are two main differences:

(i) $m_{27}$: An additional term has to be added to take into account the bending.

(ii) $m_{51}$: This has to be set to zero.

In the calculation, say for a horizontal kicker magnet, we need to add a term to the gradient $x'$ to include the bending from the reference path. Recall that for the bending magnet, the $x'$ is given by

$$x_1' = x_0' + m_{26}\delta_0 + m_{27} \tag{34}$$

Recall also that the bend angle, $\theta$, for the bending magnet is given by

$$m_{26} = \theta \tag{35}$$

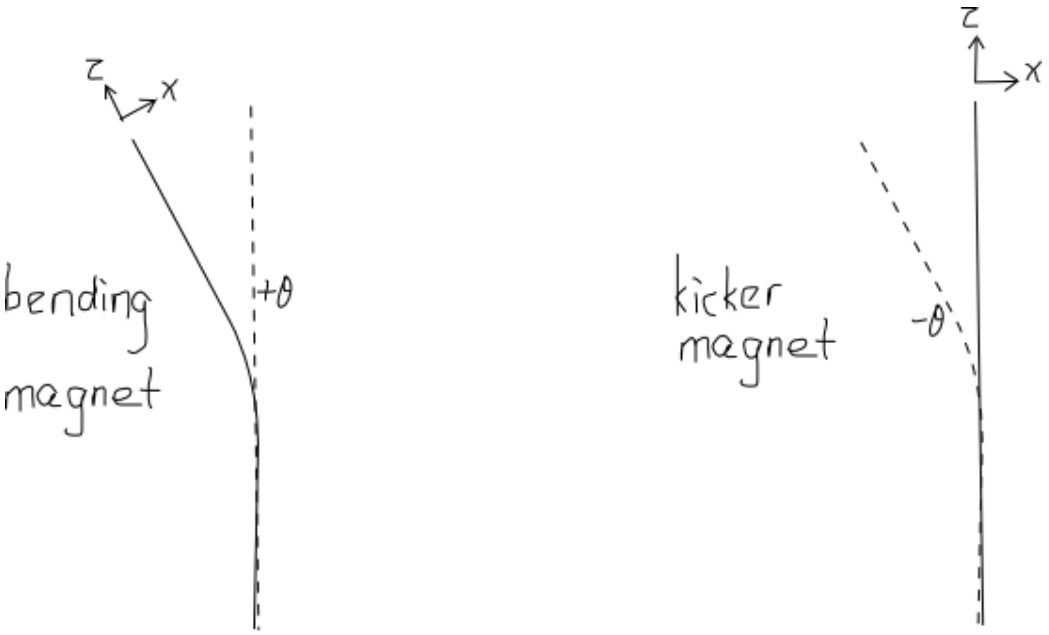


Figure 7: Bending magnet bends electrons to the design path. Kicker magnet kicks electron from the design path.

where

$$m_{26} = \left(\frac{l}{B\rho}\right) B_y \tag{36}$$

Assuming that the bend angle is small, we can just add $\theta$ (in radians) to $x'_0$. We must be careful about the signs here. In the page on bending magnets, I have implicitly assumed that if the trajectory goes straighter than the bent reference path, there would be a positive deviation in $x$.

Following this convention, we should give $\theta$ a minus sign before adding it to $x'_0$. The equation would look like this:

$$x'_1 = x'_0 + m_{26}\delta_0 + m_{27} - \theta \tag{37}$$

Remember from bending magnet that the term $m_{26}\delta_0 + m_{27}$ gives the change in bending due to energy change caused by radiation. This remains the same.

To specify the matrix elements for the kicker, we could absorb $\theta$ into $m_{27}$:

$$m_{27} = \frac{1}{2}\left[\left(\frac{l}{B\rho}\right) B_y\right] m_{67} - m_{26} \tag{38}$$

where $m_{67}$ and $m_{26}$ are as defined in bending magnets, and where I have made use of the relation that $\theta$ is equal to $m_{26}$.

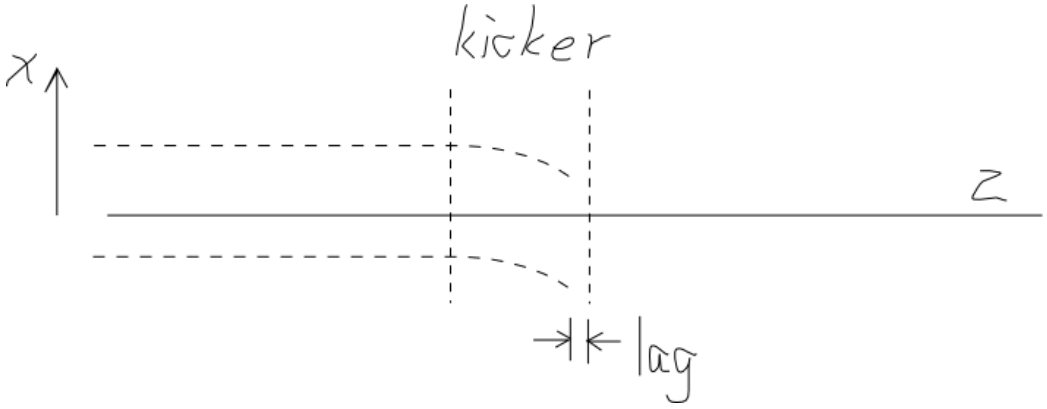


Figure 8: Deviation in $z$ is the same for any deviation in $x$.

Next, we must set $m_{51}$ to zero. Recall how this is derived in bending magnet. The relevant equation is:

$$z_1 = z_0 + m_{51}x_0 \tag{39}$$

The expression for $m_{51}$ is derived by considering the following.

When the electron is displaced horizontally from the reference path on entering the magnet, it lags behind an electron on the reference path on leaving. Because it has to bend further from the centre - fig. 6.

But this reasoning does not apply to the kicker magnet at all. The reference path of the kicker magnet is not bent!

In fact, the reference path through the kicker magnet is usually straight. If an electron comes in, we would expect it to be bent and give the same lag in z which ever side it is displaced horizontally - fig. 8.

For a small bend angle, this lag would be second order. (I cam taking the bend as a first order term.) Since I am considering only linear dynamics, this lag may be neglected, and $m_{51}$ can be set to zero.

The matrix equation is therefore given by:

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{40}$$

where

$$m_{26} = \left(\frac{l}{B\rho}\right) B_y \tag{41}$$

$$m_{67} = -C_\gamma E_0^3 \Big[\Big(\frac{l}{B\rho}\Big) B_y\Big]^2 \Big(\frac{1}{2\pi l}\Big) \quad (42)$$

$$m_{27} = \frac{1}{2}\Big[\Big(\frac{l}{B\rho}\Big) B_y\Big] m_{67} - m_{26} \quad (43)$$

Note that these equations are slightly different from those listed in [2]. $m_{27}$ is really the same - just that I have written it in another way. $m_{51}$, however, is missing.

As explained above, $m_{51}$ is only present when the reference path is bent. After computing the closed orbit the kicker transfer matrix must be replaced by the bending magnet matrix. The $m_{51}$ would be reinstated, and $m_{27}$ would revert to its original bending magnet form.

The reason again has to do with the way that the calculation is usually done. After computing the closed orbit, the coordinates of the electron are then computed in terms of its deviation from the closed orbit instead of from the original reference path. This means that the closed orbit is effectively taken as the new reference path.

This new reference path would include the bend produced by the kicker. Then we don't need to add the angle $\theta$ to the gradient $x'$ any more. The kicker can be treated as just another bending magnet, now that the strength has been fixed.

I have not actually explained how the kicker strength is fixed. There are two ways (at least) in which kickers may be treated. One way is to use it to simulate a displaced quadrupole in calculations. Another way is to put them next to quadrupoles to correct for the misalignments of the quadrupoles in experiments.

A displaced quadrupole can be expressed in terms of the sum of a quadrupole field and a dipole field. It is this dipole field that can be modeled by a kicker.

Correction of the mislaignment of the quadrupoles in experiments is a complex topic. One example is the work of my colleague in ref. [10].

Vertical kicker magnet follows very similar reasonings.

# 6 RF Cavity

In this section, I shall go through the basics of how an rf cavity is used to accelerate electrons in a damping ring.

When electrons go round the ring, it loses energy to radiation. When it passes through an rf cavity, it experiences an electric field in the forward direction. This gives it a push and replaces the energy that it has lost to radiation.

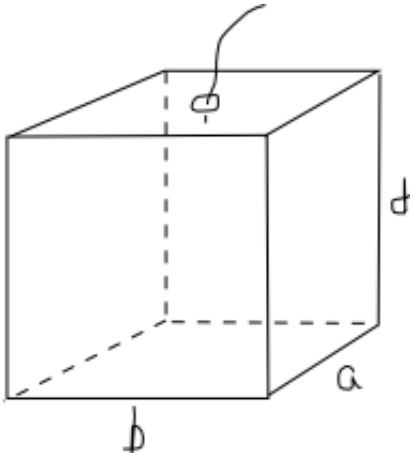


Figure 9: Current oscillating at the tip of the wire induces oscillating electric field in the metal box.

The rf cavity is basically a hollow metal box that can provide the electric field in the right direction, with the right strength, and at the right time. That is quite an impressive piece of technology. The basic idea, luckily, may be understood in a simple way.

The simplest such cavity is a metallic, rectangular box - fig. 9. Imagine making a tiny hole on one side of the box, and sticking a tiny wire through it - just a little.

We then apply an oscillating voltage to this wire. This would cause charges to move in and out of the tip of the wire sticking into the box. The electric field from these charges would then induce charges on the inner walls of the box. Depending on the frequency of the voltage, these patterns of these induced charges can get quite complicated.

We are more interested in the electric fields produced by these induced charges. We are hoping to find a way to produce a field that has a uniform direction, so that it can be used to push the electron beam in an accelerator.

If we just want a uniform field, why can't we use two parallel plates? I think it is because a closed box can produce a much stronger field. As we shall see, it has a natural frequency of oscillation for the electric fields (or induced charges). This means that the field can resonate and grow stronger and stronger - if the voltage on that wire through the hole has the same frequency.

To understand the rf cavity, we need to understand why it has a natural frequency, and what the field looks like at this frequency. The proper way to do this is to solve Maxwell's equations, and that involves some mathematics. This belongs to the topic on microwave waveguides. The details can be found

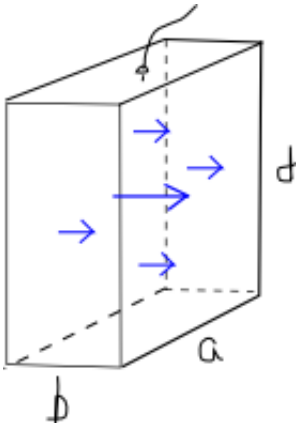


Figure 10: Choosing $b$ to be the shortest side can result in a field that is parallel to side $b$.

in textbooks like [11].

Here, I shall just quote the results from the solution. As it turns out, there are many natural frequencies. Not all frequencies are possible, however, because the electric fields must obey certain boundary conditions. For example, on the wall, the component parallel to the wall must be zero. The result is a pattern of electric fields that looks like the standing wave of a string tied to the walls at both ends. The field can only have discrete values of frequencies, given by:

$$f_{mnl} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2} \tag{44}$$

where $m$, $n$ and $l$ are integers, and $a$, $b$ and $d$ are the sides as labeled in the figure above.

Each set of $m$, $n$ and $l$ would have a particular field pattern. As it turns out, the following precription gives us what we need:

(i) Choose one side, say $b$, to be shorter than the other two.

(ii) Choose $m = 1$, $n = 0$, $l = 1$.

This choice gives an oscillating electric field that is always parallel to the side $b$ - fig. 10. It changes direction at the oscillation frequency, but must remain parallel to $b$. It is uniform if we move along direction $b$. But if we move along directions $a$ or $d$, it decreases - to zero when we reach the side walls.

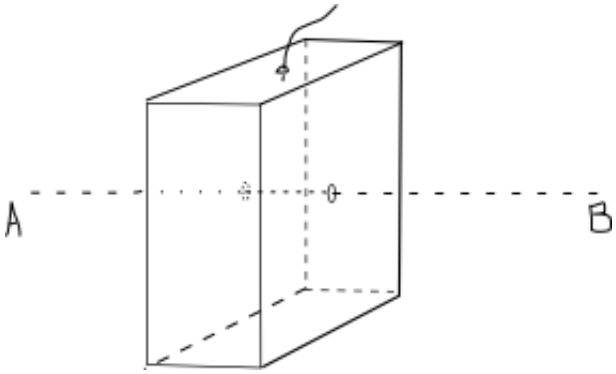


Figure 11: Making two tiny holes would not disturb the field very much.

If the voltage from the wire is maintained at $f_{101}$, the field will always look like this. It so happens that, because $b$ is the shortest side, $f_{101}$ is the lowest possible frequency for any choice of integers $m$, $n$ and $l$. In practice, this makes it easy to control the voltage frequency. It tends to be further, in terms of percentage, from other possible frequencies. If there is some error in the voltage frequency, it would not end up exciting some other frequencies, or modes. If the frequency is too high, say at $f_{201}$, the field could look quite different and may not be useful for our purpose.

So the field is maximum at the centre. If we now make two small holes on the two opposite faces perpendicular to the field, we can pass an electron into one hole, and the field would accelerate the electron to the other hole - fig. 11! So if AB is the reference path, just put the box in the path so that the path goes through the holes, and it will do the job of accelerating the particle.

There is just one little thing to worry about. The field does change direction every cycle of the oscillation. This means that it is in the right direction half the time, but the wrong direction the other half of the time. This means that we cannot have a continuous beam of electrons coming through the holes. We can only send electrons through when the direction is right, and take a break when it is not. Because of this, we would end up with separate bunches of electrons. But that is quite all right.

That's the basic idea for the rf cavity. In real accelerators of course, the shape, size and material have all been greatly improved upon. They tend to be cylindrical, have many of them stringed together, look smooth and shiny, and can even be superconducting.

Whatever it looks like and however powerful, its effect on the electron can be decribed by the same transfer matrix. I shall now derive the matrix

elements for the rf cavity. In [2], these are given as:

$$m_{65} = \frac{e\hat{V}h\cos\phi_s}{RE_0} \tag{45}$$

$$m_{67} = \frac{e\hat{V}\sin\phi_s}{E_0} \tag{46}$$

where $h$ is the harmonic number, $\hat{V}$ is the peak cavity voltage, $R$ the ring circumference, $\phi_s$ the synchronous phase, and $E_0$ the design particle energy.

All other matrix elements are zero. The matrix equation would look like this:

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{65} & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{47}$$

So there is really only one equation:

$$\delta_1 = m_{65}z_0 + m_{67} \tag{48}$$

In order to understand this equation, we must be aware of the assumption used. This is: The time spent by the electron in the cavity is much shorter than half a cycle of the electric field oscillation.

The half a cycle refers to that half when the field is in the right direction. In practice, the electron may spend a longer time. Fortunately, we just have to multiply $\hat{V}$ by some transit factor and use the same equation. I shall not go into this here.

$\hat{V}$ is maximum gain in potential of the electron when it goes through the cavity. In the simple box model above, it would be just the peak electric field magnitude multiplied by the distance $b$. For real cavities, it has to be determined by simulation softwares or measurements. I shall not worry about this here. The electric field varies sinusoidally with time. If the electron goes through the cavity, the gain in potential would depend on the phase at which it goes through.

If it enters at a phase $\phi_s$, the potential gain would be $\hat{V} \times \sin\phi_s$ - fig. 12. The change in energy would be the electron charge $e$ multiplied by this, and the fractional energy change is obtained by dividing this by the design energy $E_0$. This finally gives $m_{67}$ in Eq. 46, and contributes to the final energy change as in Eq. 48.

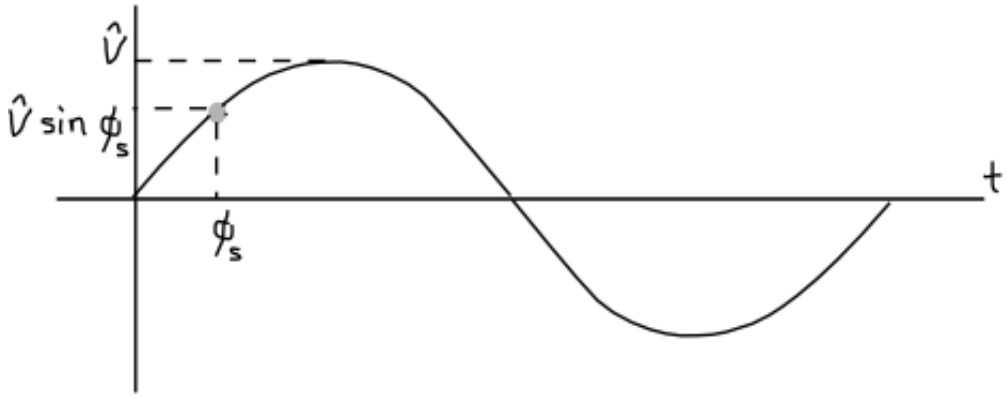


Figure 12: The energy gained by an electron in a cavity depends on the phase at which it enters the oscillating field.

The synchronous phase $\phi_s$ labeled in fig. 12, which shows how the potential gain in the cavity varies with time, would appear to be the obvious way to define it. However, for Eq. 48., $\phi_s$ is defined in another way. We may imagine a travelling wave carrying the electron around the ring - fig. 13. This electron always stay around the same phase $\phi_s$ on the wave. The potential gain in the cavity would vary sinusoidally as the wave travels through it, just as it normally does.

Now, we have a sine function with respect to $z$ instead of $t$. This makes it easier to derive the next term in Eq. 48, which involves $z_0$. First, we need an expression for the potential gain, $V$. As the electron goes round the ring, it must always enter the cavity again at the same phase of its field oscillation. This means that the time it takes to go around the ring must be a multiple of the cavity oscillation period. We may imagine that this corresponds to an integer number of wavelengths of the travelling wave around the ring. This number is the harmonic number, $h$, defined above. This means that one wavelength is equal to the ring circumference $2\pi R$ divided by $h$. This knowledge allows us to write down an expression for the travelling wave:

$$V = \hat{V}\sin(kz - \omega t) \tag{49}$$

where

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\pi R/h} = \frac{h}{R} \tag{50}$$

and $\omega$ is the angular frequency of the cavity field oscillation.

We are now ready to derive the term in Eq. 48 involving $z_0$. This term arises if the electron is slightly ahead or behind its design path, i.e. when $z_0$ is not zero. When this happens, the phase of the electron would be slightly

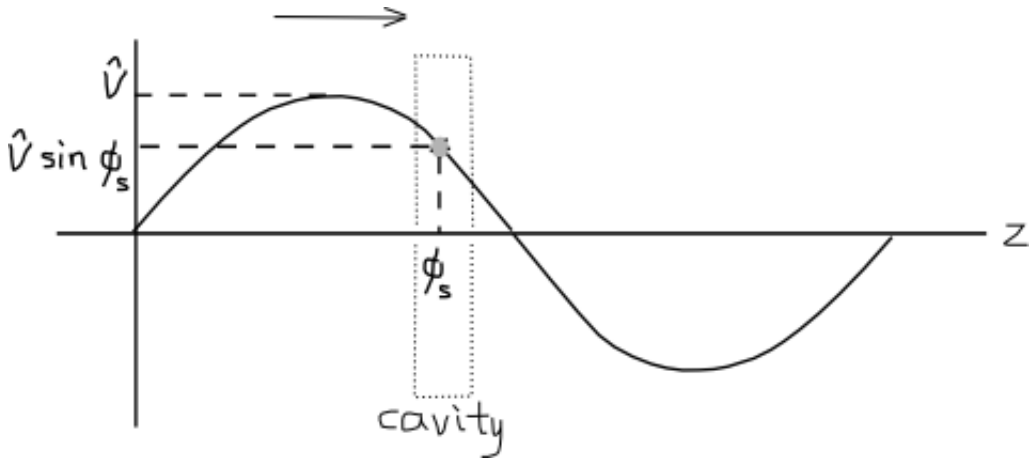


Figure 13: A "travelling" wave round the ring. This really exists only in the cavity, indicated by the dotted box.

different in fig. 13. The change in potential can be estimated using the first order approximation:

$$\delta V = k\hat{V}\cos(kz - \omega t)\delta z \tag{51}$$

The phase $(kz - \omega t)$ would be the synchronous phase $\phi_s$ at the cavity. The $\delta z$ would be $z_0$. The contribution to the fractional change in energy is then given by

$$\frac{e\delta V}{E_0} = e\frac{h}{R}\hat{V}\cos\phi_s z_0 \tag{52}$$

This gives required the term in Eq. 48.

# 7 An Interim Conclusion

We are not quite at a conclusion yet, since this is only the first of possibly two or three parts. However, it is becoming clear that the initial aim to develop a simple, intuitive and unifying view of damping ring physics is developing well. Other elements such as quadruples and sextupoles can clearly be described in a similar way. It remains to demonstrate in subsequent parts that these basic elements are sufficient to produce features such as closed orbits, dispersion and frequency maps.

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