Liverpool HEP Seminars,
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# Lattice QCD

**Andreas Jüttner**

UNIVERSITY OF
Southampton

# Outline

1. Lattice QCD (why and what)
2. Precision Flavour Physics
3. $(g\text{-}2)_\mu$ on the lattice
4. Pushing the Frontiers (QED+QCD, rare decays)

# Motivation

- Standard Model of elementary particle physics describes electromagnetic, weak and strong (QCD) interactions consistently in terms of a renormalisable quantum field theory
- but there is substantial phenomenological evidence that it can't be the whole story: dark matter, CP-violation, … indicate that there must be sth. else
- despite decades of experimental and theoretical efforts we have not found a smoking gun

# Motivation

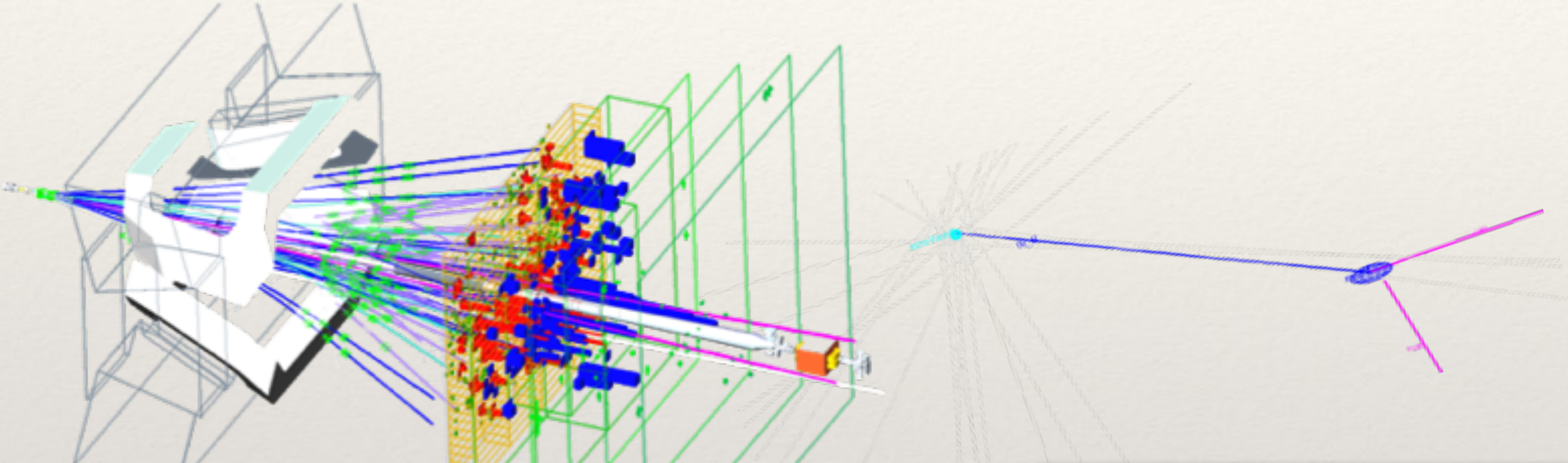
- searches for new physics: direct vs. indirect search:
  - ‘bump in the spectrum’
  - SM provides correlation between processes experiment + theory to over-constrain SM

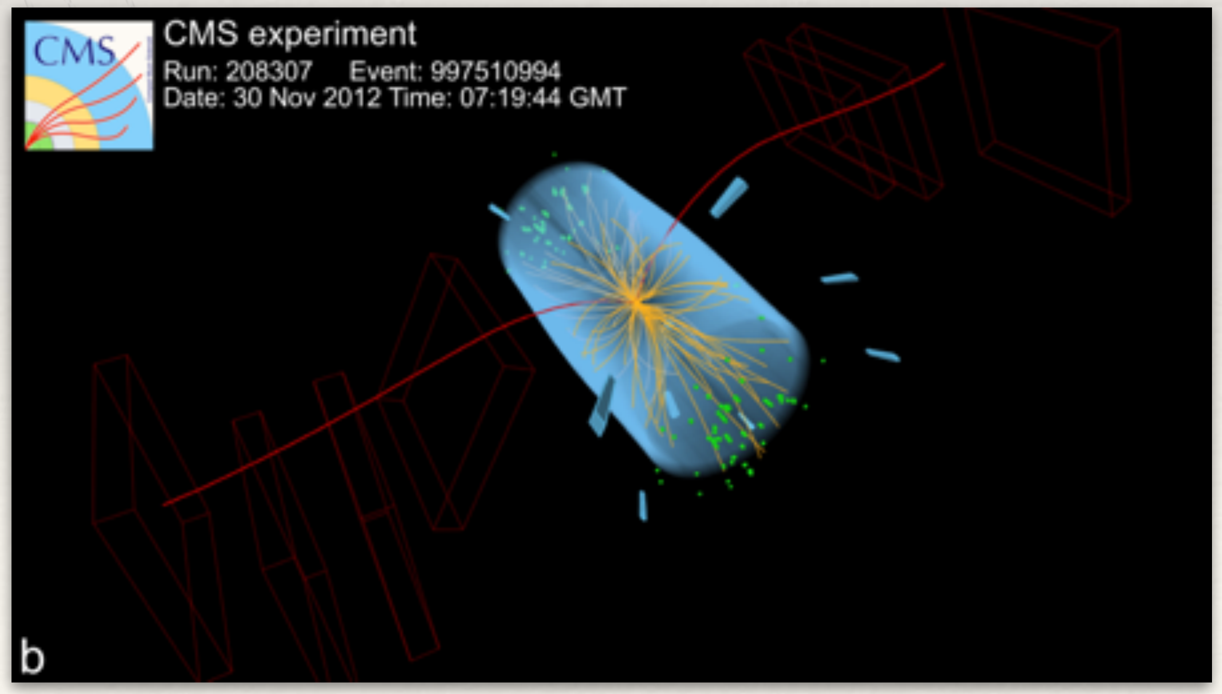
- hadronic (QCD) uncertainties dominating error budget

- lattice QCD can in principle provide the relevant input and is becoming increasingly precise in its predictions

# $B_s \rightarrow \mu^+\mu^-$

**First observed by LHCb, CMS**





# $B_s \to \mu^+\mu^-$

## Standard Model prediction:

- Loop suppressed in the SM (FCNC) → sensitive to non-SM interaction?

$$\mathrm{Br}(B_s \to \ell^+\ell^-)^{\mathrm{SM}} = \tau_{B_s} \frac{G_F^2\alpha^2}{16\pi^3} |V_{tb}V_{ts}^*|^2 m_{B_s} m_\ell^2 \beta_\ell(m_{B_s}^2) |C_{10}|^2 \boxed{f_{B_s}^2}$$

$$Br \propto (PT) \times \boxed{\langle 0|\bar{s}\gamma_\mu\gamma_5 b|\bar{B}_s\rangle^2} + \dots$$

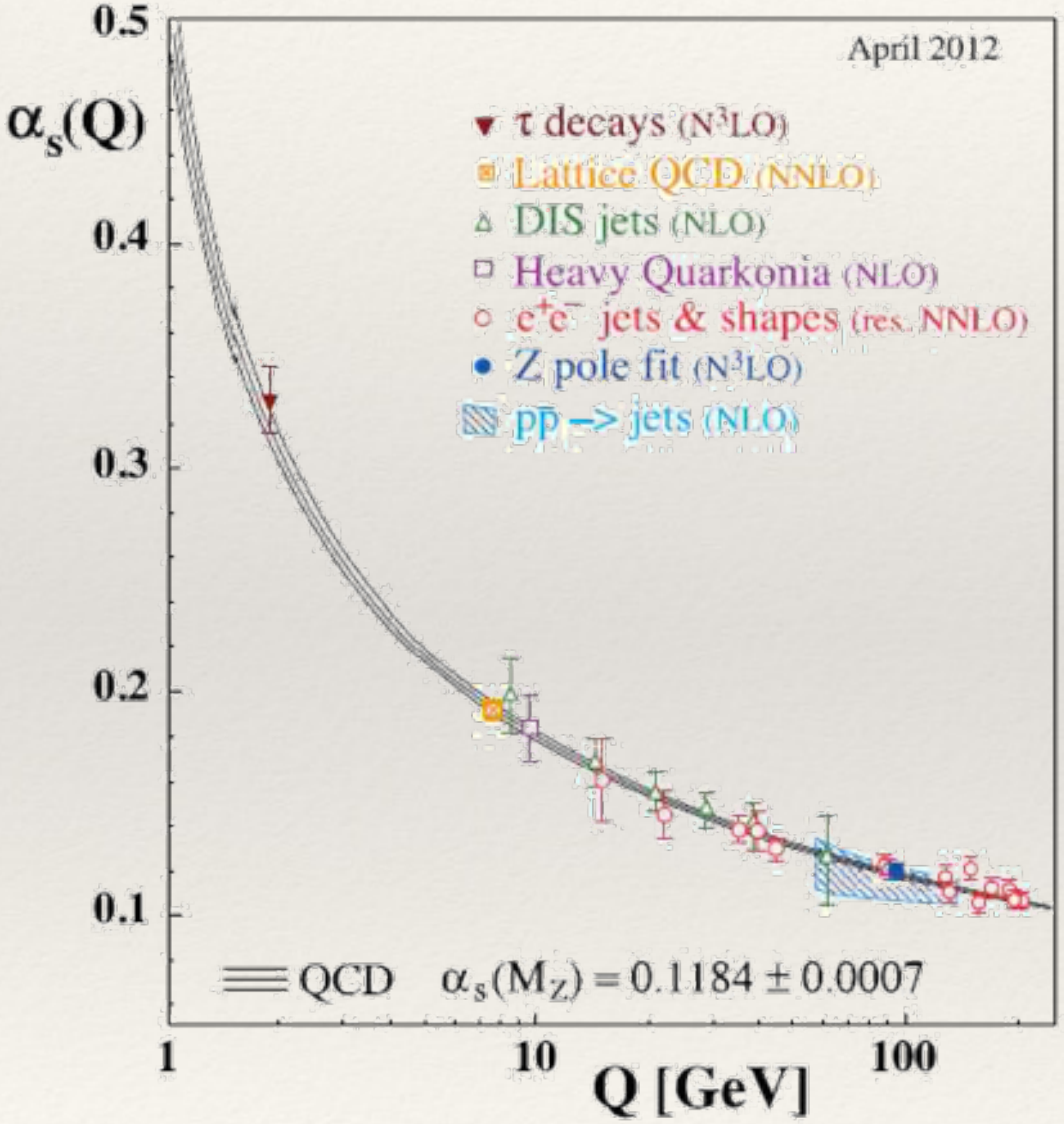
NNLO QCD
NLO EW

Hermann, Misiak, Steinhauser, JHEP 1312, 097 (2013)
Bobeth, Gorbahn. Stamou, PRD 89, 034023 (2014)

very precise and reliable prediction for the decay constant is needed

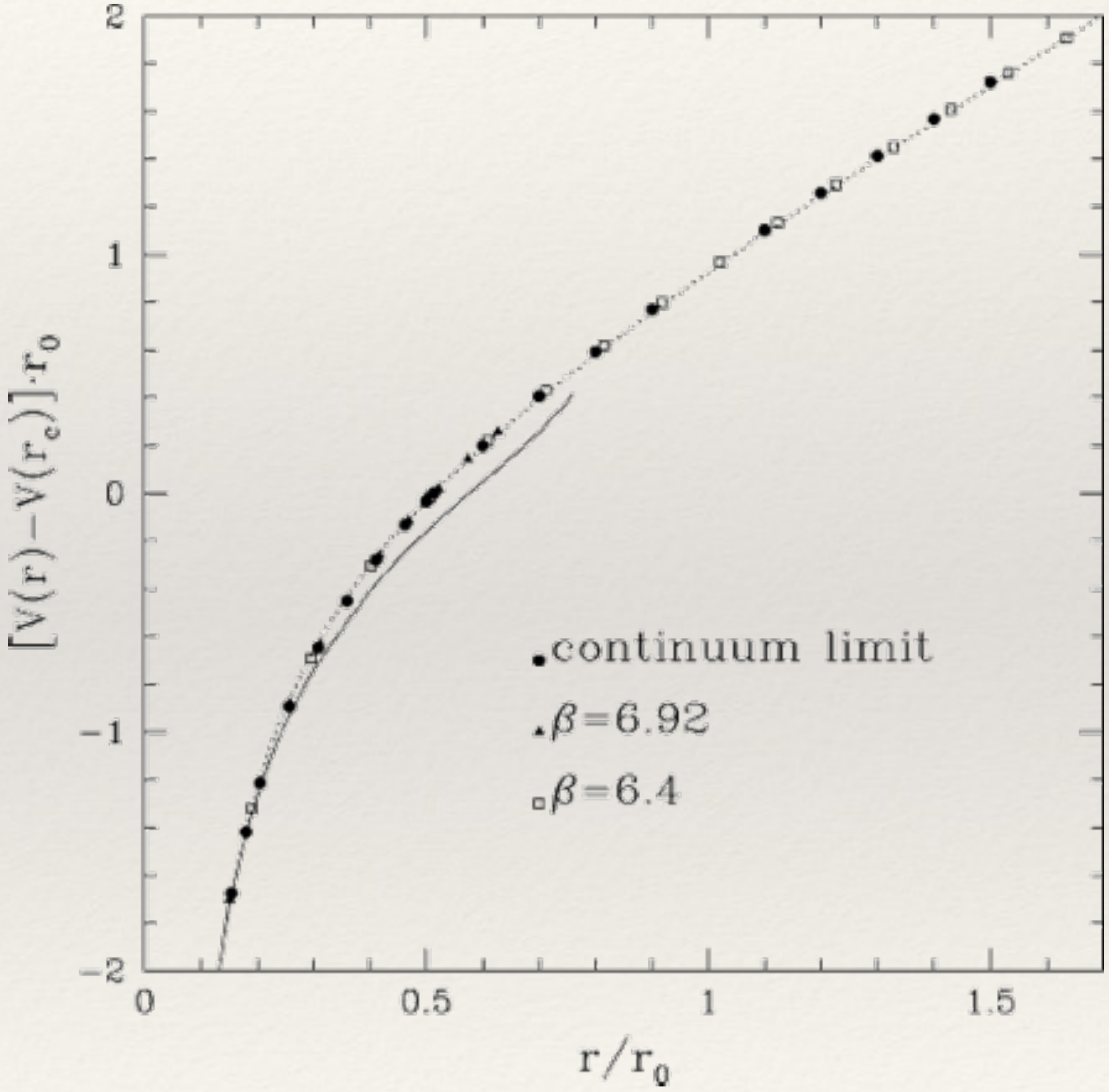
# QCD

## asymptotic freedom



PDG

## confinement



Necco & Sommer NPB 622 (2002)

# Lattice QCD

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F^{a}_{\mu\nu}F^{a\,\mu\nu} + \sum_{f}\bar{\psi}_f\left(i\gamma^{\mu}D_{\mu} - m_f\right)\psi_f$$

Free parameters:

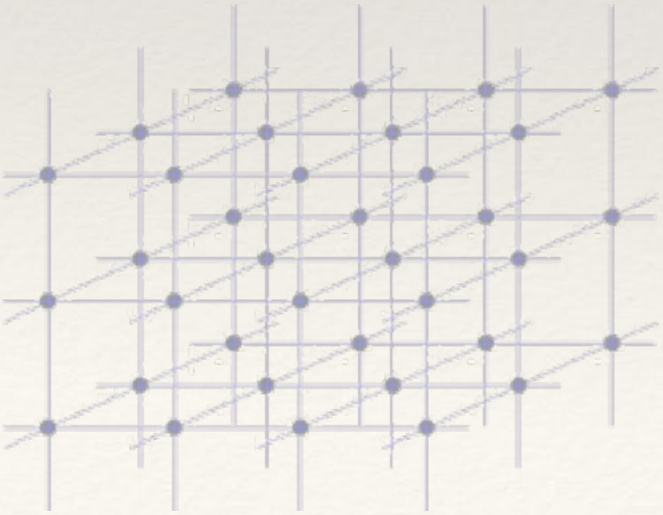
- gauge coupling $g \rightarrow \alpha_s = g^2/4\pi$
- quark masses $m_f = u,d,s,c,b,t$

- Lagrangian of massless gluons and *almost massless quarks*
- what experiment sees are bound states, e.g. $m_\pi, m_P \gg m_{u,d}$
- underlying physics non-perturbative

Path integral quantisation:

$$\begin{aligned}\langle 0|O|0\rangle &= \tfrac{1}{\mathcal{Z}}\int\mathcal{D}[U,\psi,\bar{\psi}]\,O\,e^{-iS_{\text{lat}}[U,\psi,\bar{\psi}]}\\ \langle 0|O|0\rangle &= \tfrac{1}{\mathcal{Z}}\int\mathcal{D}[U,\psi,\bar{\psi}]\,O\,\boxed{e^{-S_{\text{lat}}[U,\psi,\bar{\psi}]}}\end{aligned}$$

Euclidean space-time
Boltzmann factor



finite volume, space-time grid (IR and UV regulators)
$\propto L^{-1}$ $\propto a^{-1}$

→ well defined, finite dimensional Euclidean path integral
→ from first principles

# Lattice QCD

- gauge-invariant regularisation (Wilson 1974)
- naively: replace derivatives by finite differences, integrals by sums
- finite volume lattice path integral still over large number of degrees of freedom > $O(10^{10})$
- Evaluate discretised path integral by means of Markov Chain Monte Carlo on state-of-the-art HPC installations

# Euclidean correlation function

$$\langle 0|\mathcal{O}_{B_s}(t)\mathcal{O}_{B_s}(0)^\dagger|0\rangle = \tfrac{1}{Z}\int \mathcal{D}[\bar\psi,\psi,U]\mathcal{O}_{B_s}(t)\mathcal{O}_{B_s}(0)^\dagger e^{-S[\bar\psi,\psi,U]}$$

$$\begin{aligned}\langle 0|\mathcal{O}_{B_s}(t)\mathcal{O}_{B_s}(0)^\dagger|0\rangle &= \sum_{\vec{x},n}\langle 0|\mathcal{O}_{B_s}(x)|n\rangle\langle n|\mathcal{O}^\dagger_{B_s}(0)|0\rangle \\ &= \sum_{n}\left|\langle 0|\mathcal{O}_{B_s}(0)|n\rangle\right|^2 e^{-E_n t_x} \\ &\overset{t\to\infty}{=} \left|\langle 0|\mathcal{O}_{B_s}(0)|B_s\rangle\right|^2 e^{-m_{B_s} t_x}\end{aligned}$$

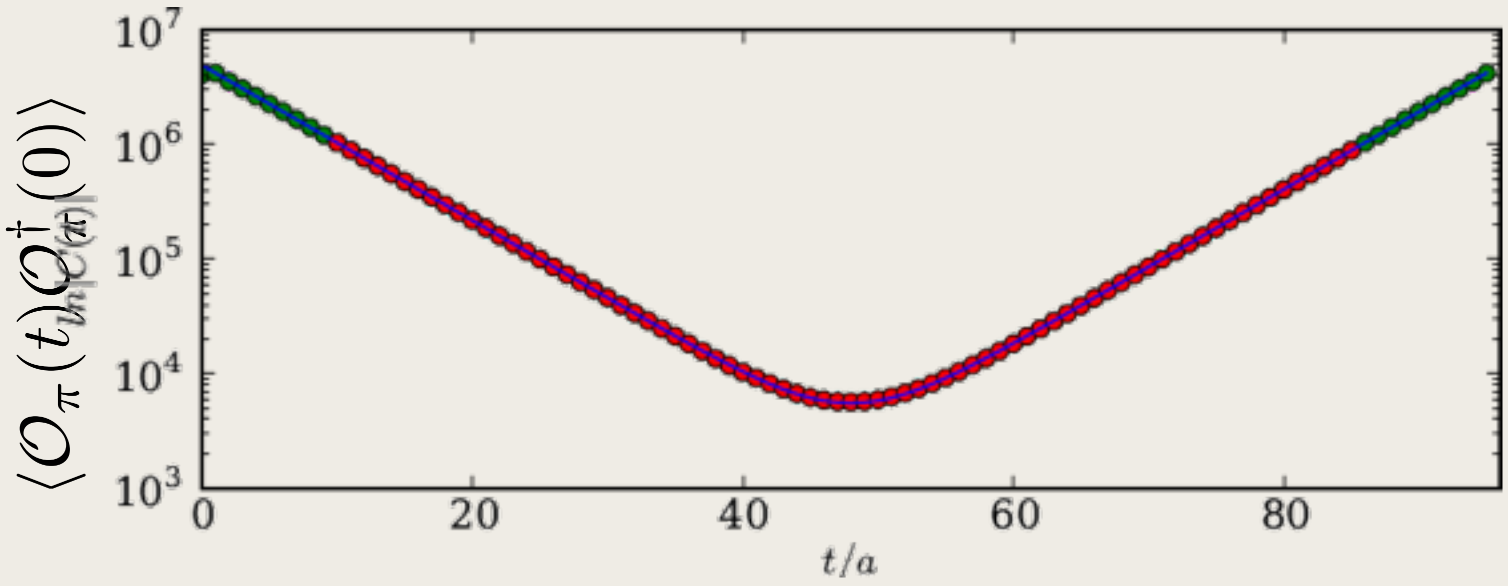
**two-point function**

extract physical properties from fits to simulation data:

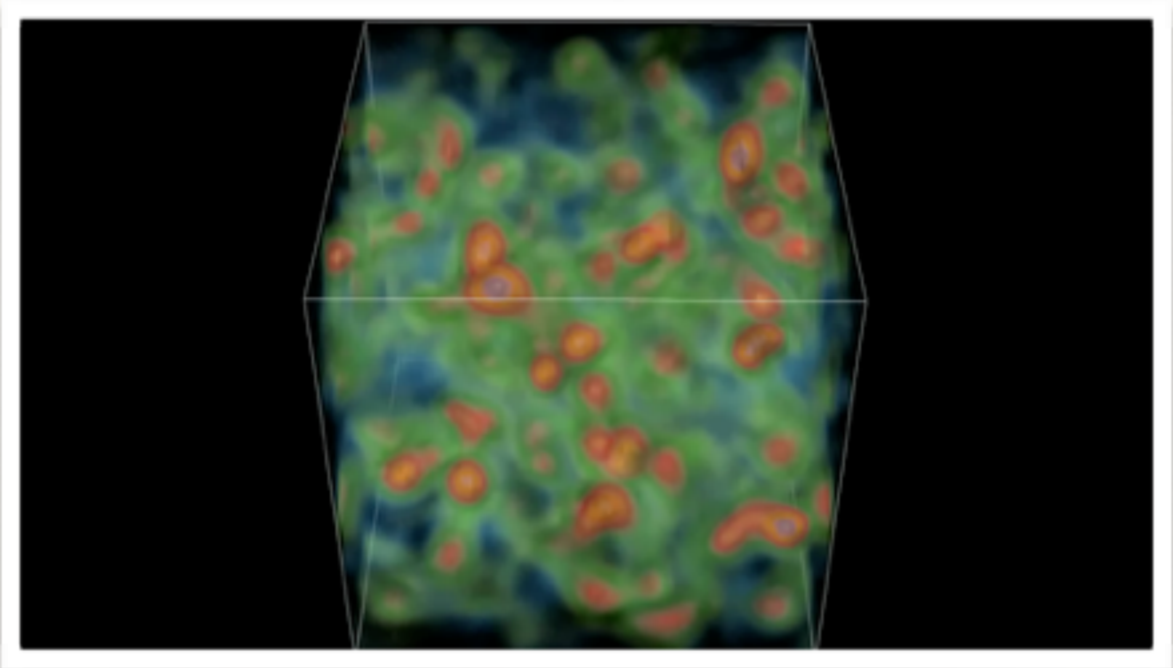
- normalisation → matrix element (e.g. decay constant)
- time-dependence → particle spectrum (e.g. meson mass)
- stat. errors from MC sampling over N field configurations

$$\langle \mathcal{O}\mathcal{O}^\dagger\rangle = \frac{1}{N}\sum_{n=1}^{N}\left[\mathcal{O}\mathcal{O}^\dagger\right]_n$$

(bootstrap, jackknife error analysis, autocorrelation analysis, …)



# State of the art of lattice QCD simulations



action density of RBC/UKQCD physical point DWF ensemble

## What we can do

- simulations of QCD with dynamical (sea) *u,d,s,c* quarks with masses as found in nature ➔ $N_f = 2,\ 2+1,\ 2+1+1$
- bottom only as valence quark
- cut-off $a^{-1} \leq 4\mathrm{GeV}$
- volume $L \leq 6fm$

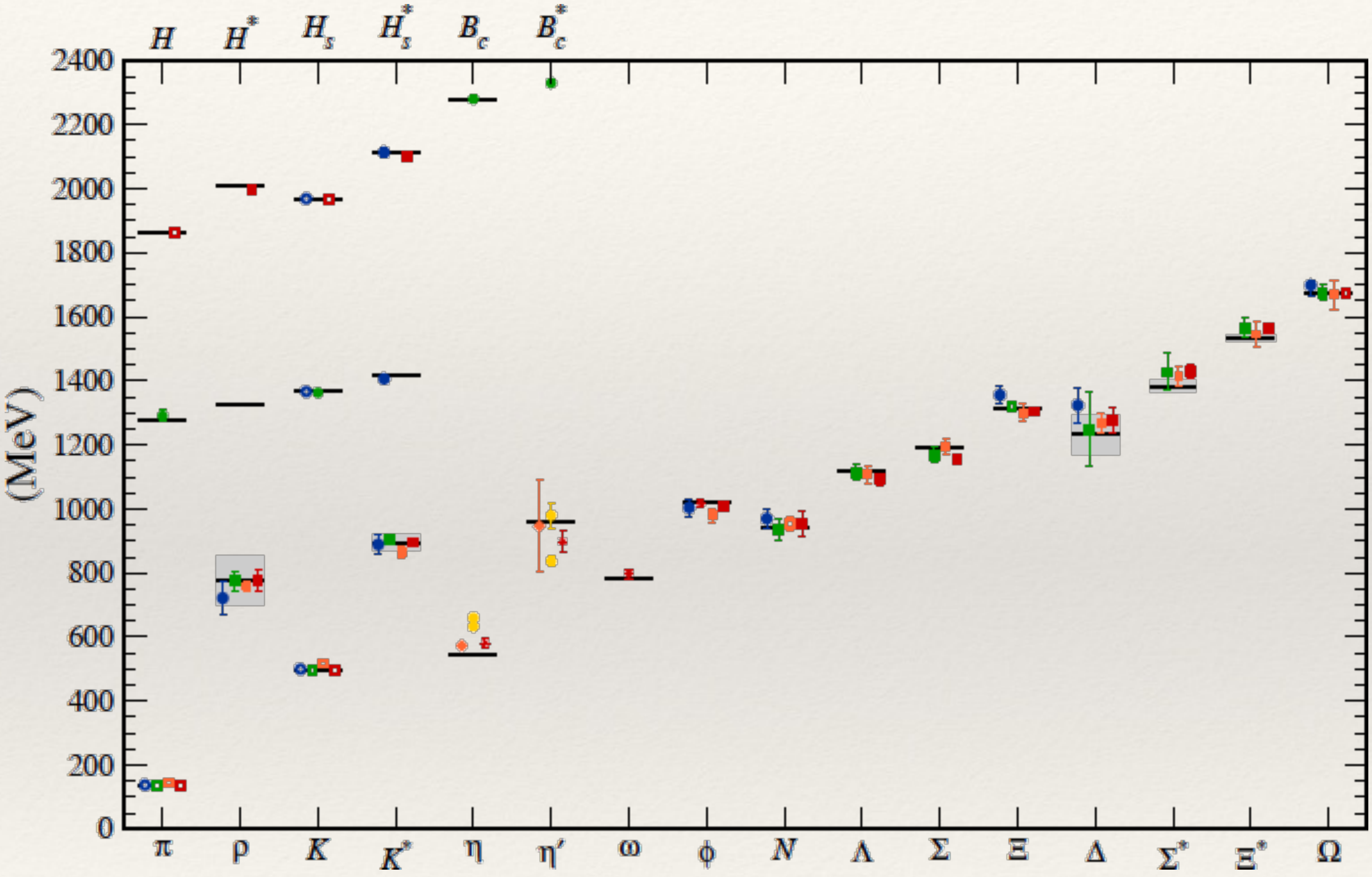
## Parameter tuning

start from *educated guesses* and compute

- tune light quark mass *am*$_l$ such that $\frac{am_\pi}{am_P} = \frac{m_\pi^{PDG}}{m_P^{PDG}}$
- tune strange quark mass such that $\frac{am_\pi}{am_K} = \frac{m_\pi^{PDG}}{m_K^{PDG}}$
- determine physical lattice spacing $a = \frac{af_\pi}{f_\pi^{PDG}}$

**IMPORTANT:**
once the QCD-parameters are *tuned* no further parameters need to be fixed and we can make fully predictive simulations of QCD

# benchmark - the hadron spectrum

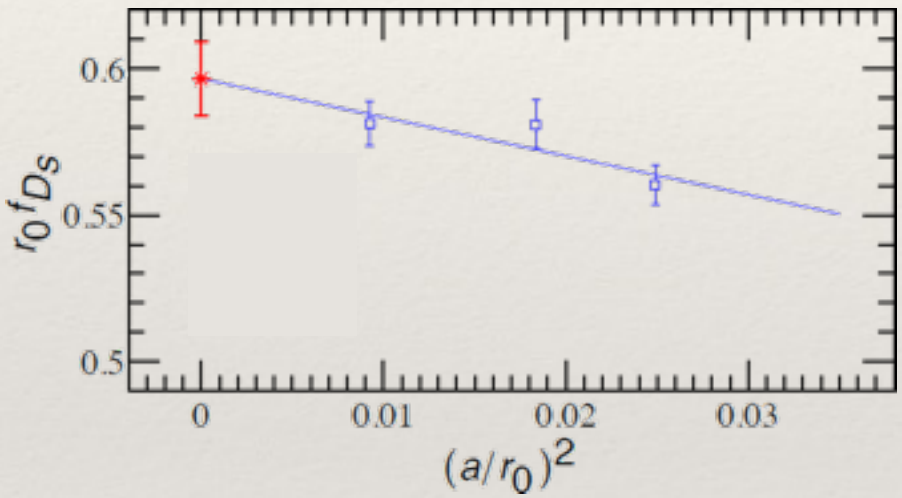


Kronfeld, Ann. Rev. of Nucl. Part. Sci 2012 62

# lattice - systematics

In practice one needs to control a number of sources of systematic uncertainties, most notably:

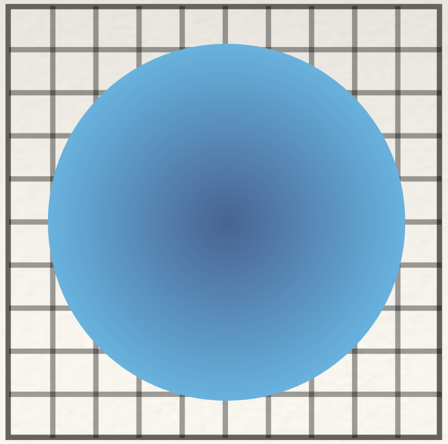
- **discr. errors** (lattice spacing *a*)



$$S_{\text{eff}} = \int d^4x\,\{\mathcal{L}_0(x) + a\mathcal{L}_1(x) + a^2\mathcal{L}_2(x) + \dots\}$$

Symanzik 1982,1983

- **finite volume errors** (box size *L*)



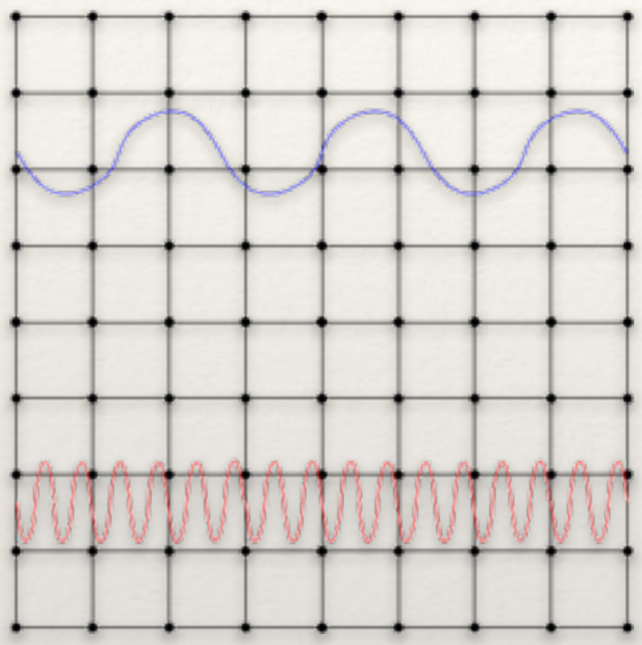
In QCD for simple ME $\propto e^{-m_\pi L} \propto O(1\%)$

more complicated for processes with several hadrons in initial or final state

Lüscher Commun.Math.Phys. 105 (1986) 153-188, Nucl. Phys. B354, 531 (1991)

quite crude, in practice more complicated

# a state-of-the-art lattice



need to keep

$a^{-1} \ll$ relevant scales $\ll L^{-1}$

- for $m_\pi$=140MeV the constraint for controlled finite volume effects of $m_\pi L \gtrsim 4$ suggests L≈6fm
- for charm quarks to be well resolved $am_c < 1$ *e.g.* $a^{-1}$ larger than ≈2.5GeV needed
- lattices with $L/a \gtrsim 80$ needed

Fulfilling all the constraints is just starting to happen
(e.g. first $96^3 \times 192$ have been generated (MILC)) in the meantime most collaborations

- weaken the finite volume effects by simulating unphysical heavy pions
- extrapolate from coarser lattices relying on assumptions for functional form of cutoff effects

# Lattice pheno - what's possible

- **Standard:**
  - meson ME with single incoming and/or outgoing pseudo-scalar states $\pi, K, D_{(s)}, B_{(s)} \to \mathrm{QCD-vacuum}, \pi \to \pi, K \to \pi, D \to K, B \to \pi, ..., B_K, (B_D), B_B$
  - QCD parameters: quark masses, strong coupling constant
  - meson/baryon spectroscopy of stable (in QCD) states

- **Challenging:**
  - two initial/final hadronic states, one channel $\pi\pi \to \pi\pi, K\pi \to K\pi, K \to \pi\pi, \ldots$
  - elm. effects in spectra
  - long-distance contributions in e.g. rare Kaon decays, $K$-mixing

- **Very challenging - new ideas needed/no clue:**
  - multi-channel final states (hadronic $D$, $B$) (e.g. Hansen, Sharpe PRD86, 016007 (2012))
  - transition MEs with unstable in/out states (Briceño et al. arXiv:1406.5965)
  - elm effects in matrix elements

# Comment: what's the problem with multi-hadron state?

scattering: contrary to single particle states there is a continuum of scattering states

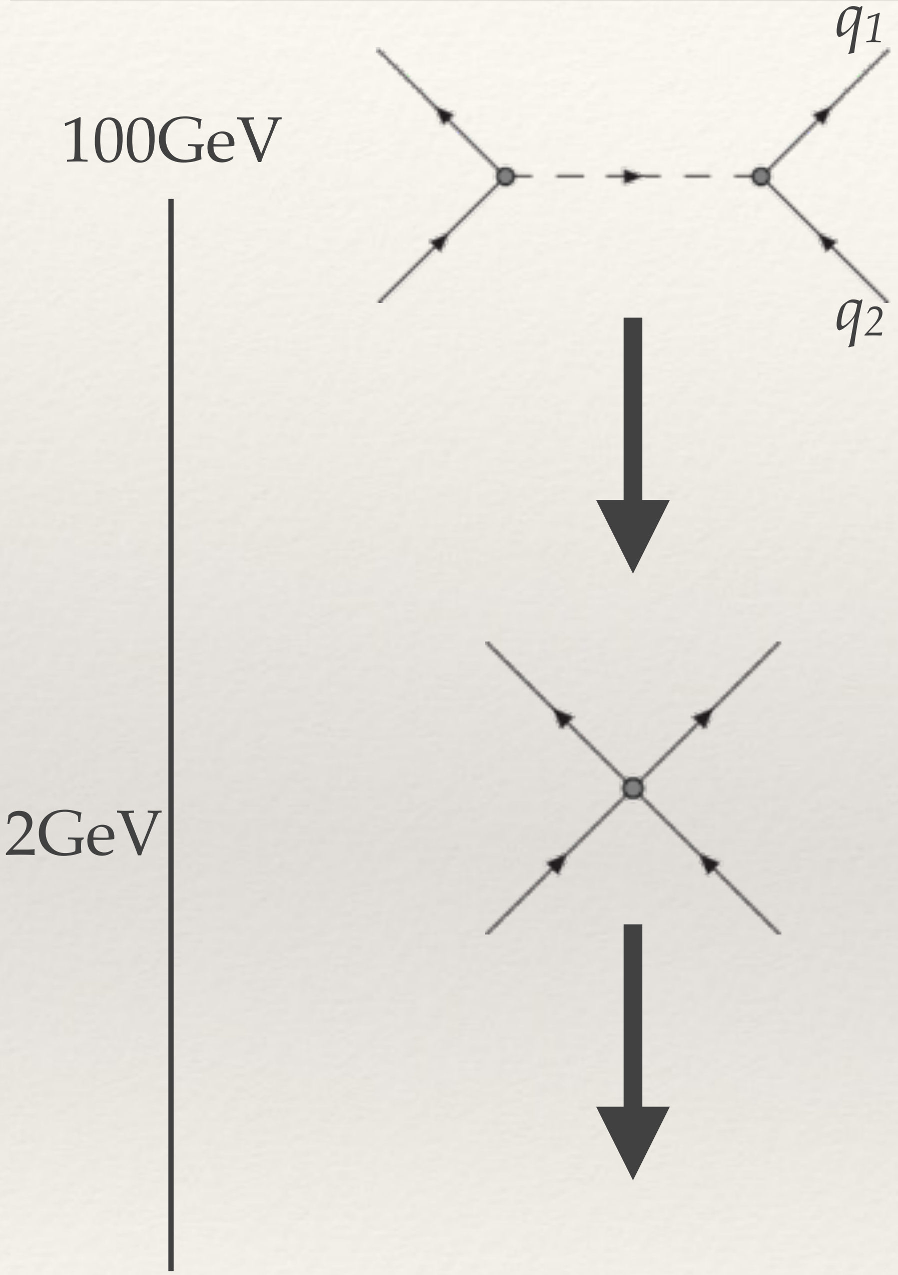
Maiani-Testa-No-Go: can't extract scattering energies and ME from Euclidean $n$-pt functions

However: in finite volume scattering energies still discrete and they can be related analytically to the scattering phase shift

**Lüscher, NPB354 (1991) 531-578, NPB364 (1991) 237-254**

in practical applications mostly limited to particles with very limited number of QCD-decay channels

# Quark Flavour Physics



$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

3x3 unitary matrix
4 unknown parameters

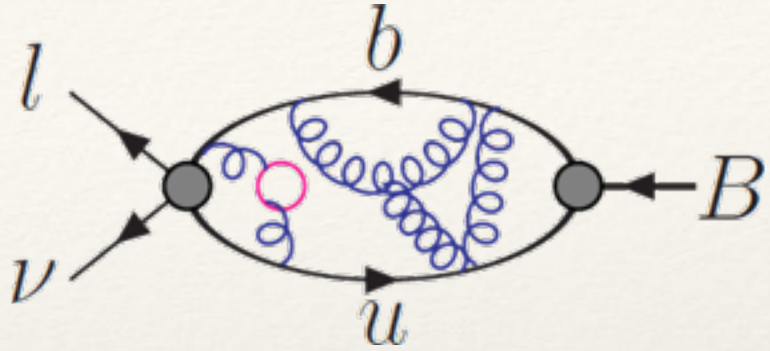
- quark mixing
- CP-violation (one complex phase)
- constraints on SM processes
- high energy reach
- inconsistencies -> failure of the SM?

300MeV $\langle \mathrm{had}_f | H_W | \mathrm{had}_i \rangle$

$\langle 0 | H_W | \mathrm{had}_i \rangle$

# Quark Flavour Physics

e.g tree level leptonic $B$ decay:



Assumed factorisation: $\Gamma_{\text{exp.}} \overset{???}{=} V_{\text{CKM}}(\text{WEAK})\underset{\text{currently EFT}}{\boxed{(\text{EM})}}(\text{STRONG})$

$$\underbrace{\Gamma(B \to l\nu_l)}_{\text{theory}} = \underbrace{|V_{ub}|^2}_{\text{output}} \underbrace{\frac{m_B}{8\pi} G_F^2 m_l^2 \left(1 - \frac{m_l^2}{m_B^2}\right)^2 f_B^2}_{\text{theory prediction}}$$

Experimental measurement + theory prediction allows for extraction of CKM MEs

# Flavour Physics

**Determine CKM elements ⟷ (indirect) test of SM:**

- over-determine elements of $V_{CKM}$ and check consistency of CKM paradigm
- unitarity tests:
  - rows and columns are (in SM) complex unit vectors
  - rows (columns) are orthogonal to other rows (columns)

  violation of unitarity would indicate non-SM physics

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \overset{?}{=} 1$$

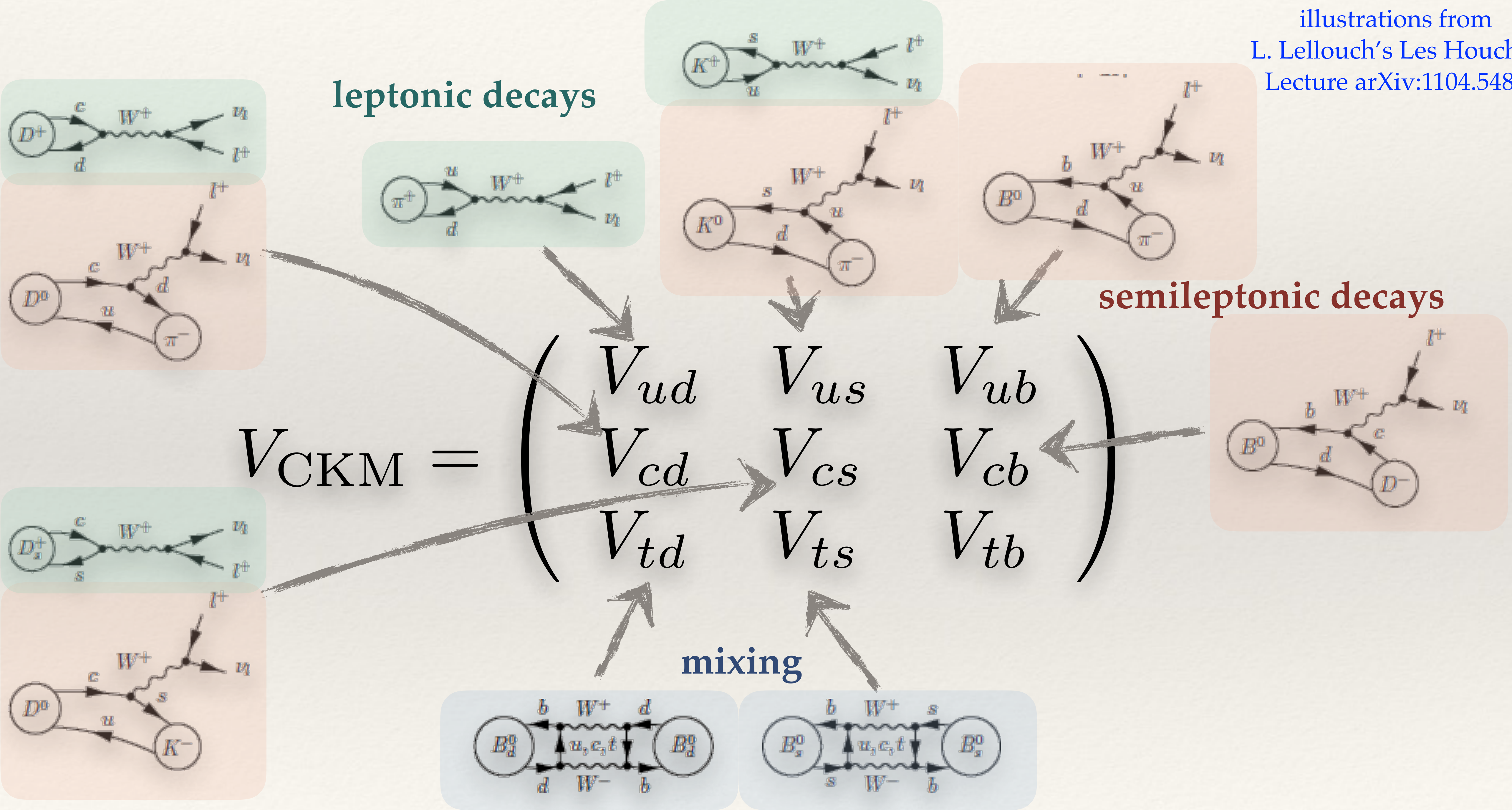
row-test

$$\sum_{U=u,c,t} V_{Ud} V_{Ub}^* = 0$$

triangle-test

- in which channels is there still room for new physics
  - how much new physics would be compatible with measurements
  - what would be the properties of new physics

# Lattice flavour physics and CKM



**leptonic decays**

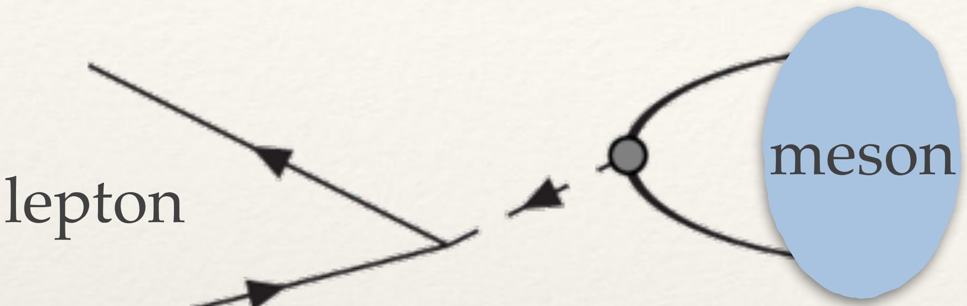
**semileptonic decays**

**mixing**

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

illustrations from
L. Lellouch's Les Houches
Lecture arXiv:1104.5484

# “tree” kaon/pion decays



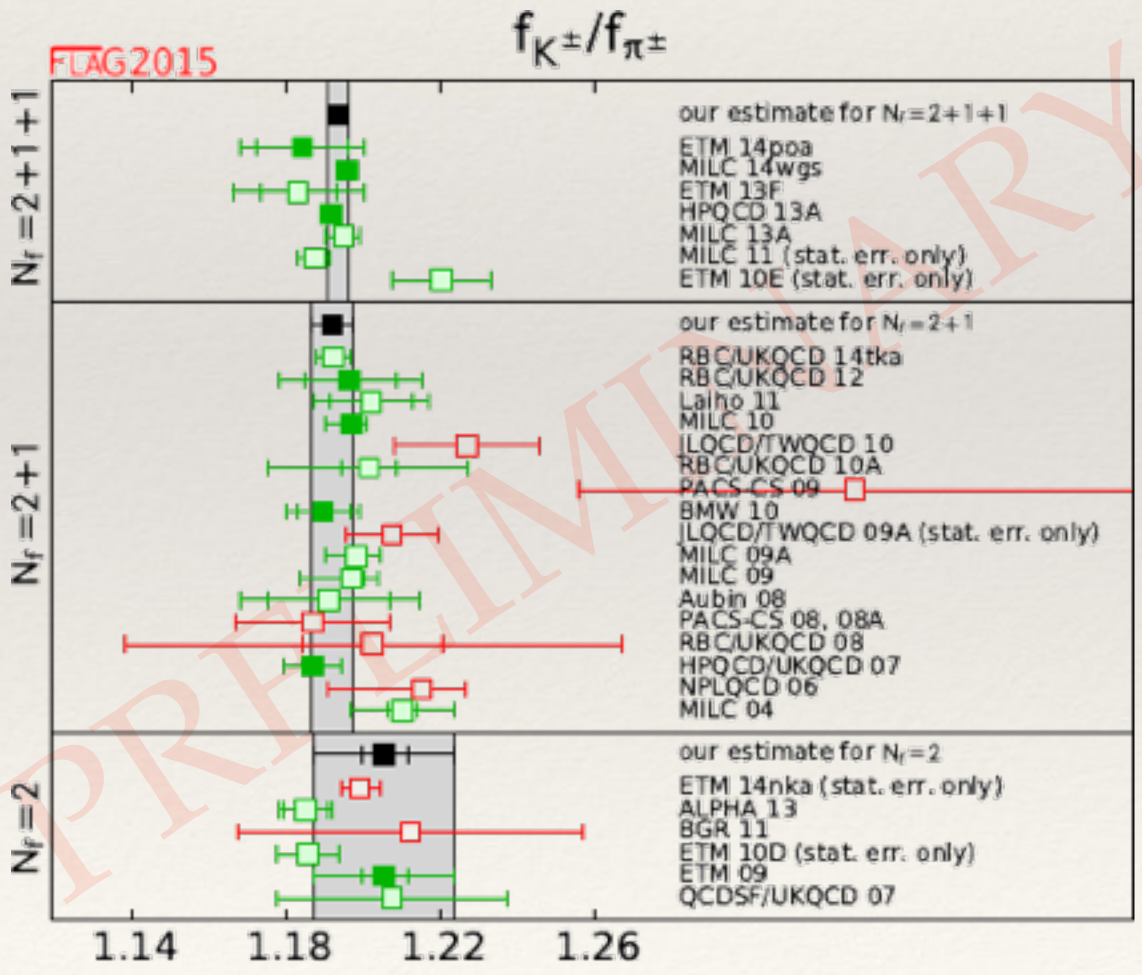
**leptonic decay**

$$\Gamma(K \to \mu\bar{\nu}_\mu) = \frac{G_F^2}{8\pi} f_K^2 m_\mu^2 m_K \left(1 - \frac{m_\mu^2}{m_K^2}\right)^2 |V_{us}|^2$$

$$\langle 0|\bar{s}/\bar{d}\gamma_\mu\gamma_5 u|K/\pi(p)\rangle = i f_{K/\pi} p_\mu$$

$$\frac{\Gamma(K\to\mu\bar{\nu}_\mu)}{\Gamma(\pi\to\mu\bar{\nu}_\mu)} = \frac{|V_{us}|^2}{|V_{ud}|^2}\left(\frac{f_K}{f_\pi}\right)^2 \frac{m_K(1-m_\mu^2/m_K^2)^2}{m_\pi(1-m_\mu^2/m_\pi^2)^2} \times 0.9930(35)$$

Marciano, Phys.Rev.Lett. 2004



1‰!!!

# Standard calculations and results - FLAG

**Flavour Lattice Averaging Group**

**"What's currently the best lattice value for a particular quantity?"**

**FLAG-1** **(Eur. Phys. J. C71 (2011) 1695)**

**FLAG-2** **(http://itpwiki.unibe.ch/flag/, Eur.Phys.J. C74 (2014) 2890)**

**FLAG-3 - working on it**

- quantities:

$$m_{u,d,s,c,b}$$

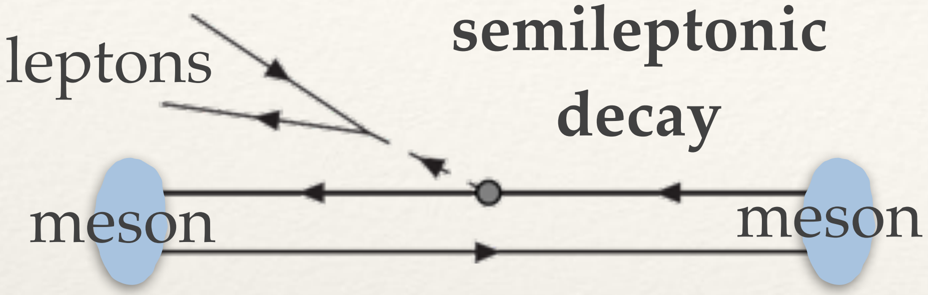
$$f_K/f_\pi,\, f_+^{K\pi}(0),\, B_K,\, SU(2) \text{ and } SU(3) \text{ LECs}$$

$$f_{D_{(s)}},\, f_{B_{(s)}},\, B_{B_{(s)}},\, B_{(s)}-\text{ and } D_{(s)}-\text{semileptonics}$$

$$\alpha_s$$

- summary of results
  - evaluation according to FLAG quality criteria (colour coding)
  - averages of best values where possible
  - detailed summary of properties of individual simulations

# “tree” kaon/pion decays



leptons

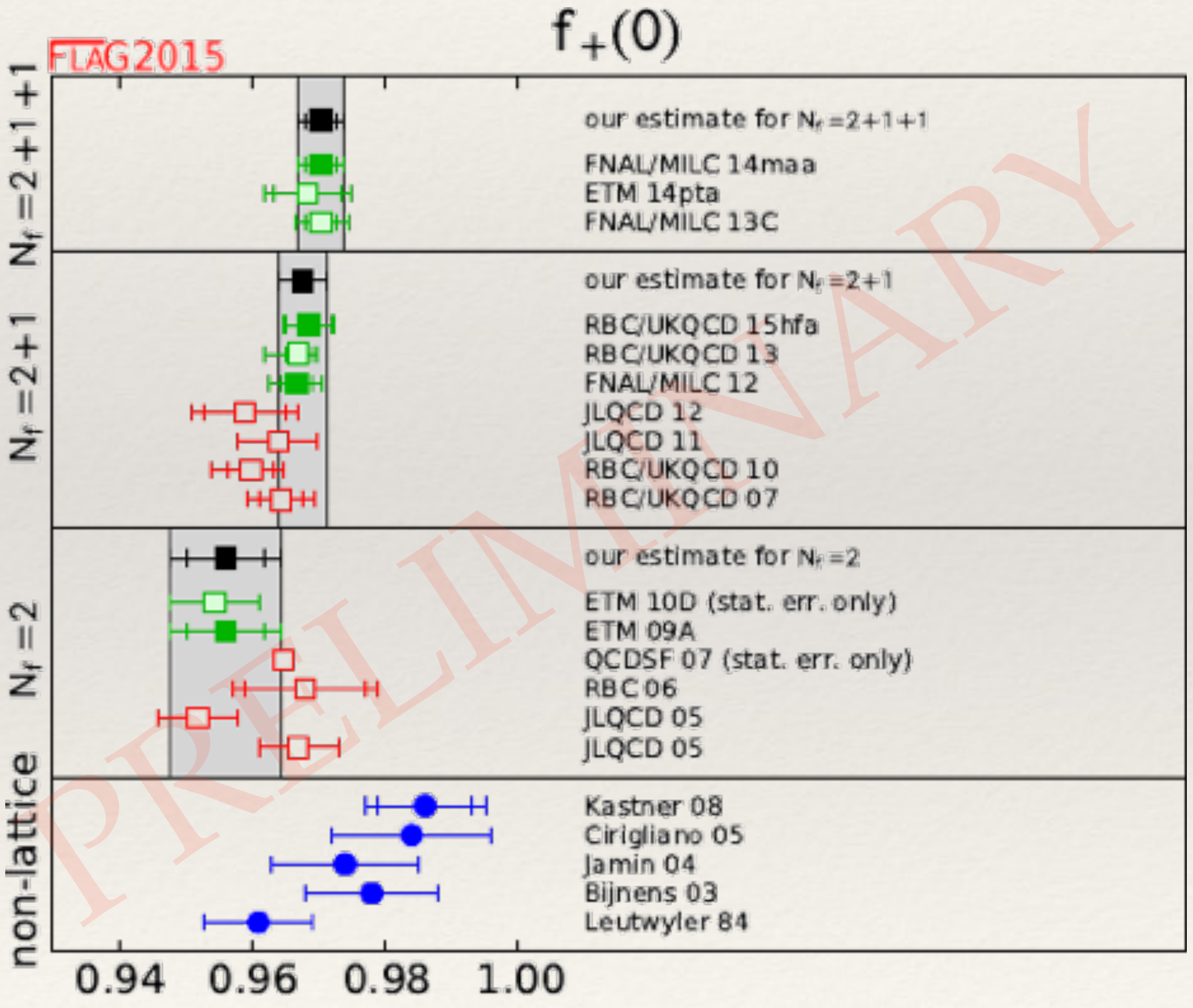
**semileptonic decay**

meson

meson

$$\Gamma_{K\to\pi l\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^2} S_{\rm EW}(1+\Delta_{SU(2)}+\Delta_{\rm EM})^2\, I\, |f_+^{K\pi}(0)|^2 |V_{us}|^2$$

$$\langle \pi(p_\pi)|V_\mu(0)|K(p_K)\rangle = f_+^{K\pi}(q^2)(p_K+p_\pi)_\mu + f_-^{K\pi}(q^2)(p_K-p_\pi)_\mu$$



**3‰!!!**

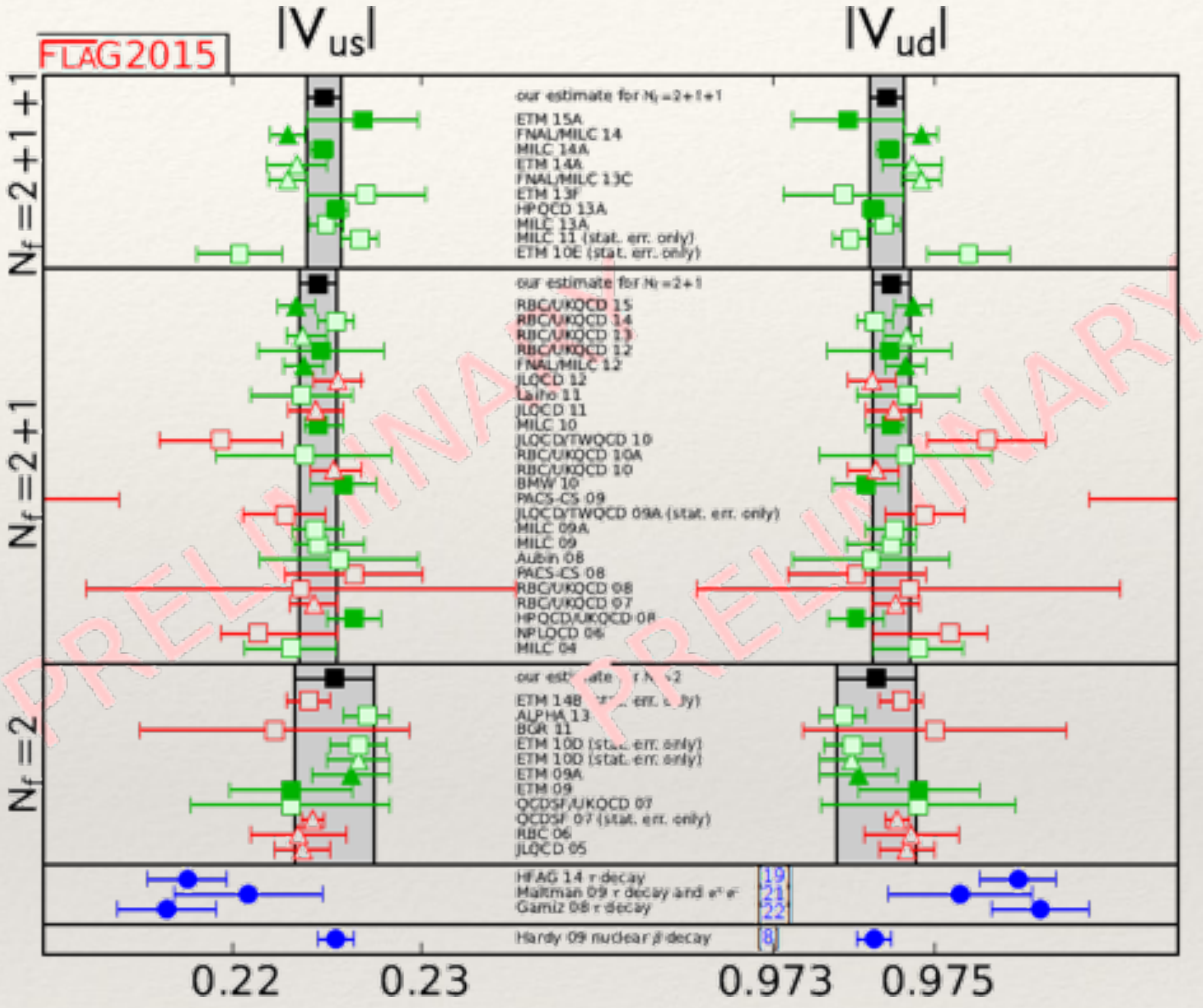
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \approx |V_{ud}|^2 + |V_{us}|^2 \overset{?}{=} 1$$

## Experimental results:

$$|V_{us}| f_+(0) = 0.2163(5)$$

$$\frac{f_K}{f_\pi} \frac{|V_{us}|}{|V_{ud}|} = 0.2758(5)$$

FLAVIA Kaon WG EPJ C 69, 399-424 (2010)
KTeV, Istra, KLOE



Numerical results from FLAG2, illustrations (preliminary) from FLAG3

## First row unitarity:

- $f_+^{K\pi}(0)$ and $|V_{ud}|$ from experiment
- $f_K/f_\pi$ and $|V_{ud}|$ from experiment
- $f_+^{K\pi}(0)$ and $f_K/f_\pi$ from lattice

|  | $f_+(0)$, $\lvert V_{ud}\rvert$ | $f_K/f_K$, $\lvert V_{ud}\rvert$ | combined |
|---|---|---|---|
| $N_f$=2+1 | 0.9993(5) | 1.0000(6) | 0.987(10) |
| $N_f$=2 | 1.0004(10) | 0.9989(16) | 1.029(35) |

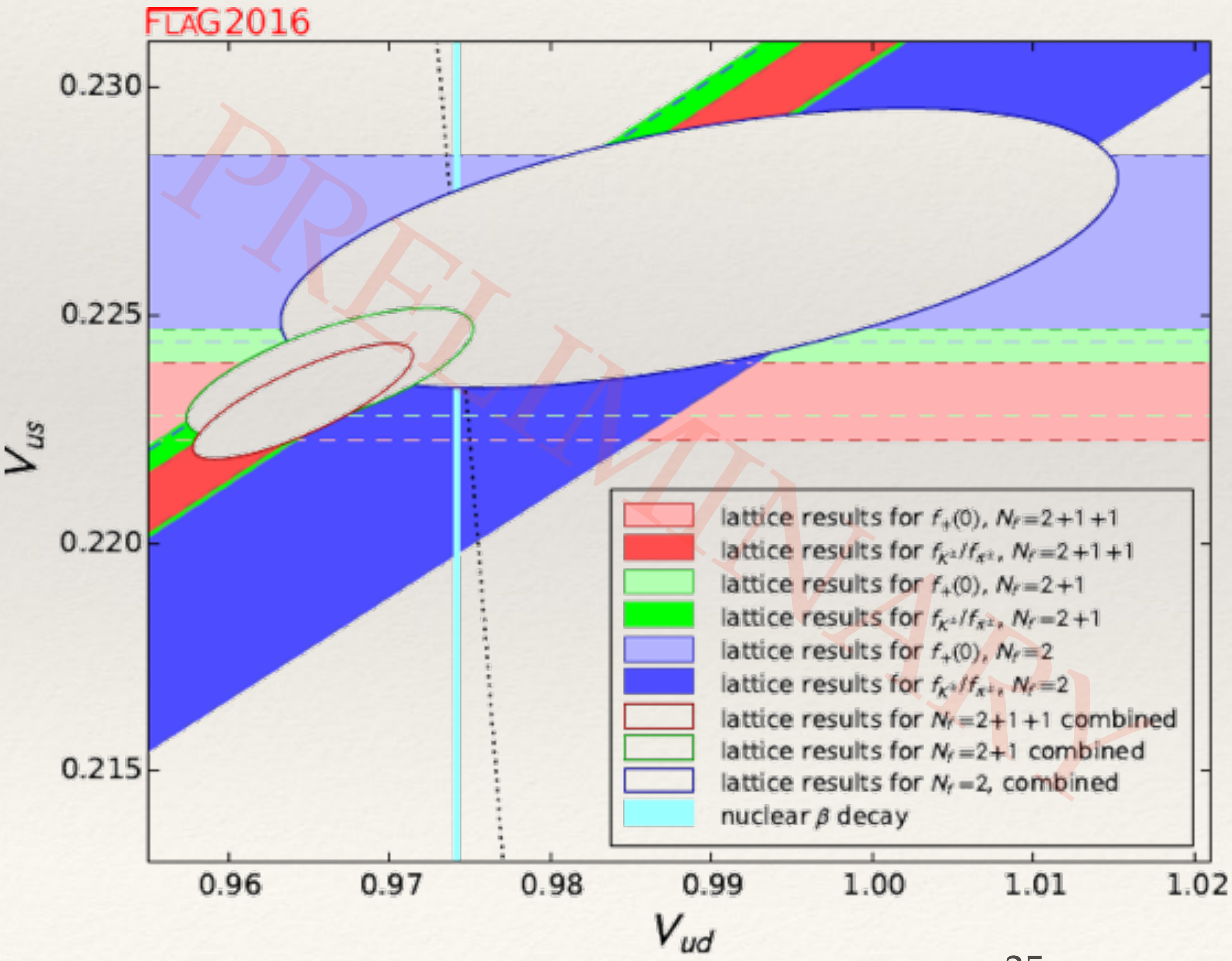
Eur.Phys.J. C74 (2014) 2890
arXiv:1310.8555

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \approx |V_{ud}|^2 + |V_{us}|^2 \overset{?}{=} 1$$

FLAG $V_{us}$ Working Group (Boyle, Kaneko, Simula)

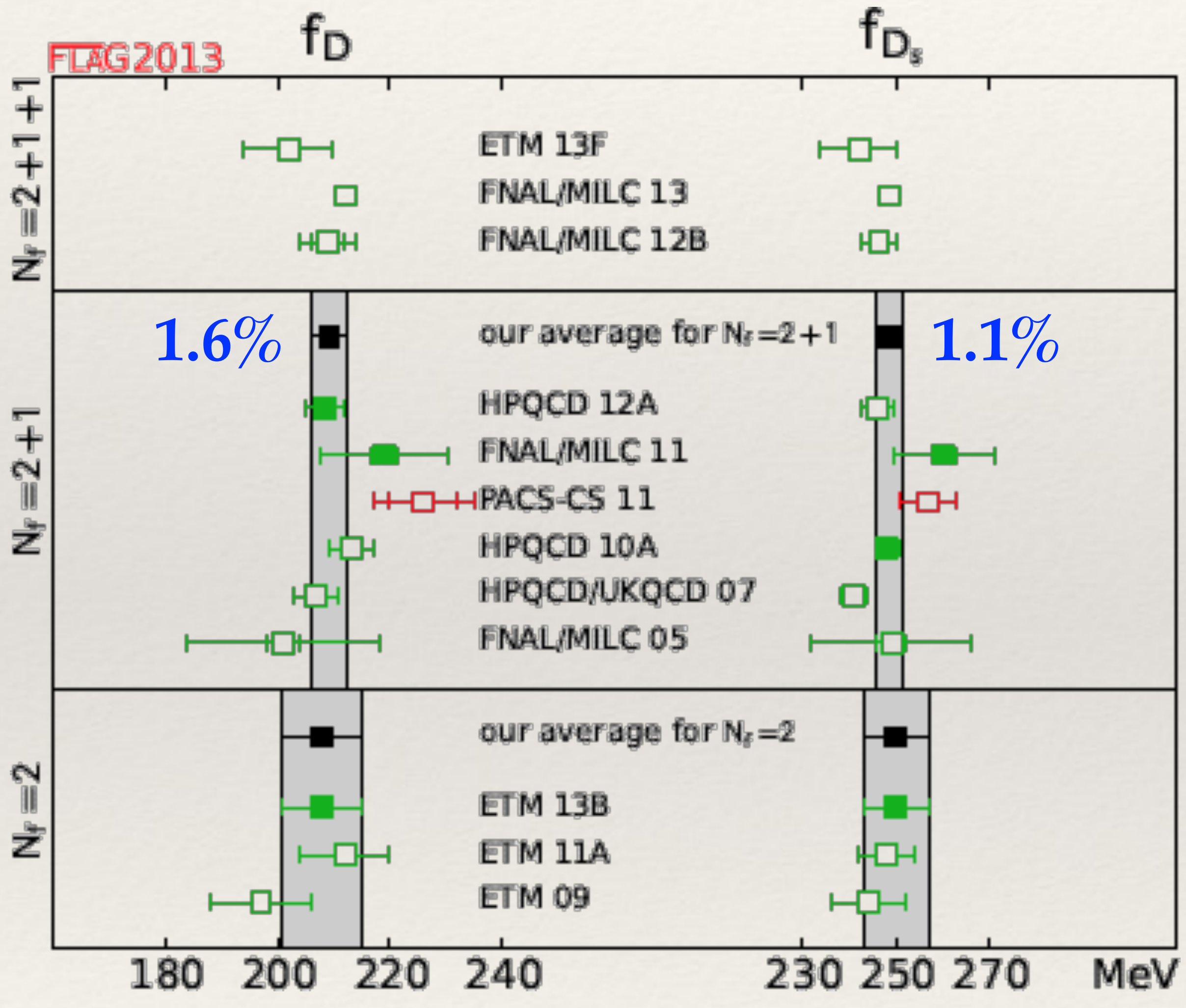
$$|V_{us}| f_+^{K^0\pi^-}(0) = 0.2163(5) \qquad \frac{f_{K^+}}{f_{\pi^+}} \frac{|V_{us}|}{|V_{ud}|} = 0.2758(5)$$

FLAVIANet Kaon WG
EPJ C 69, 399-424 (2010)
**arXiv:1005.2323**



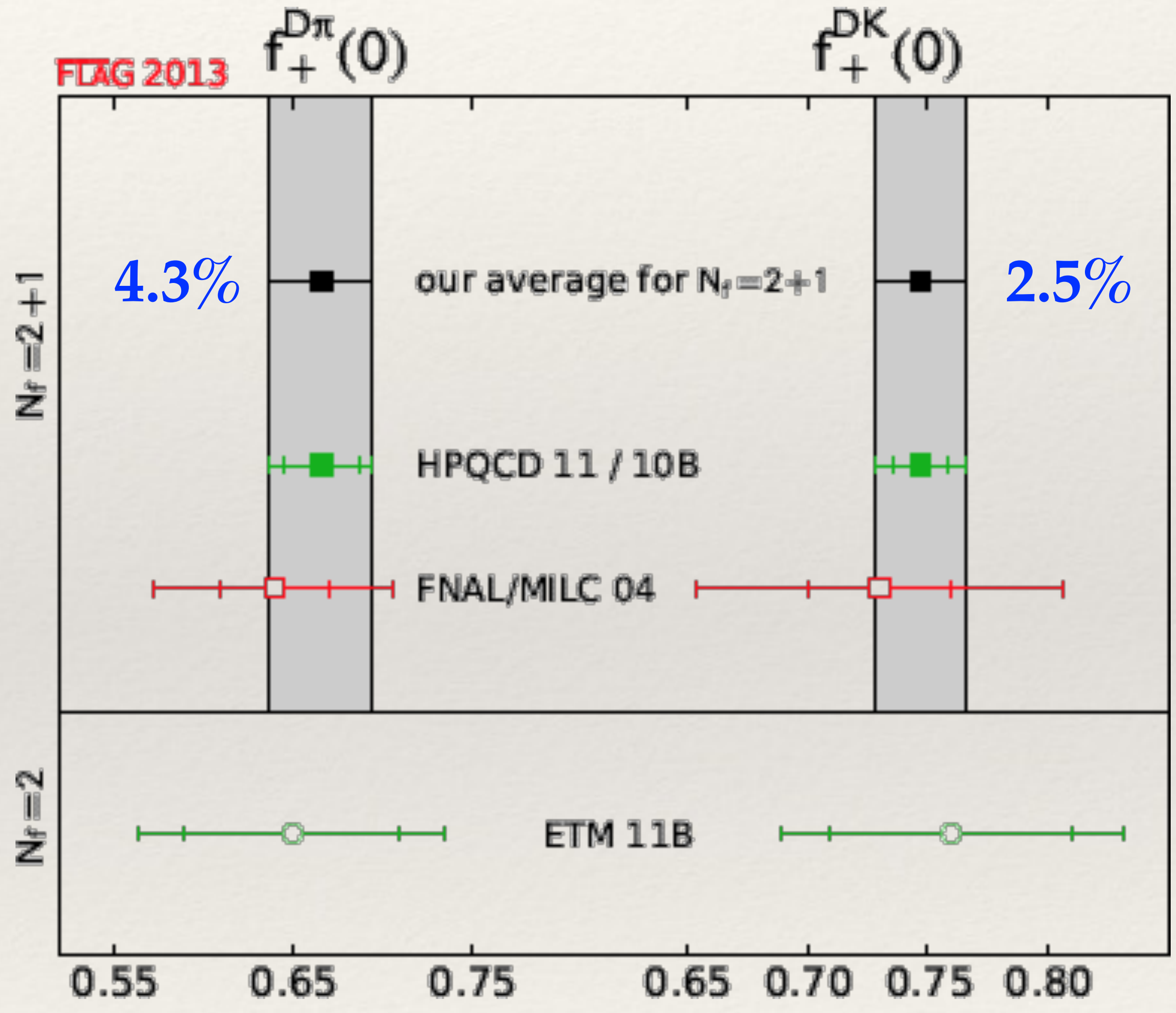
high precision test of SM unitarity - no worrisome tension at sub-percent-level precision

# Leptonic $D_{(s)}$ meson decays



1.6%

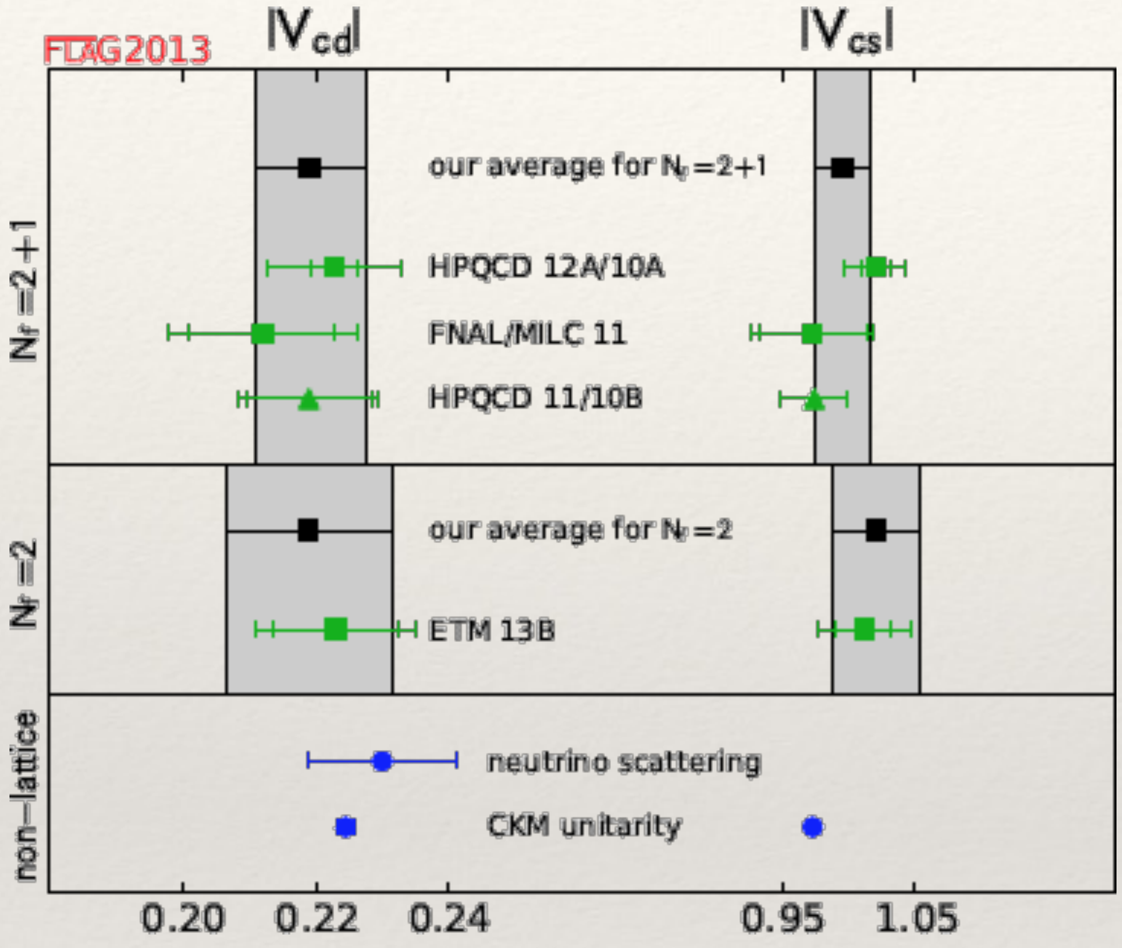
1.1%



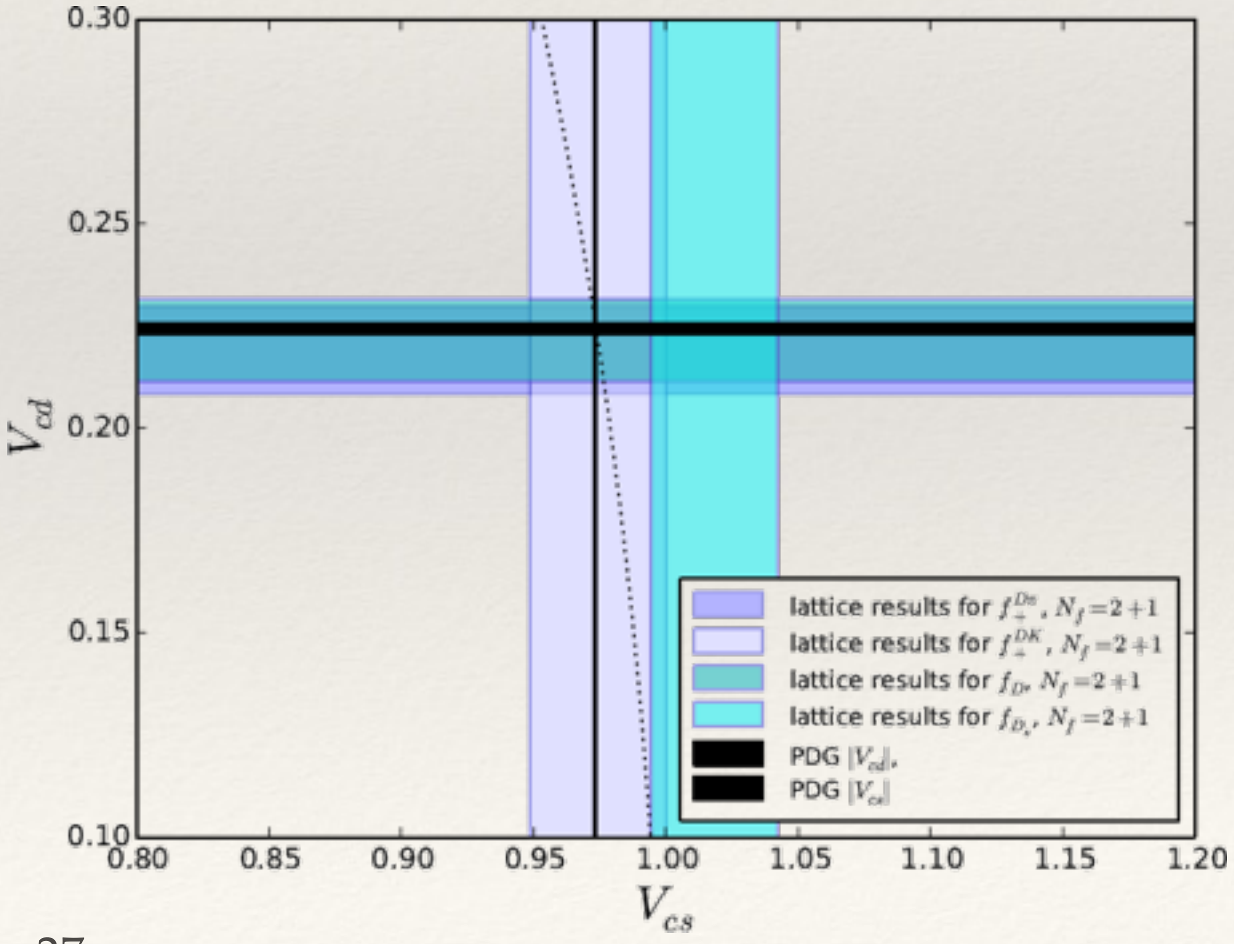
4.3%

2.5%

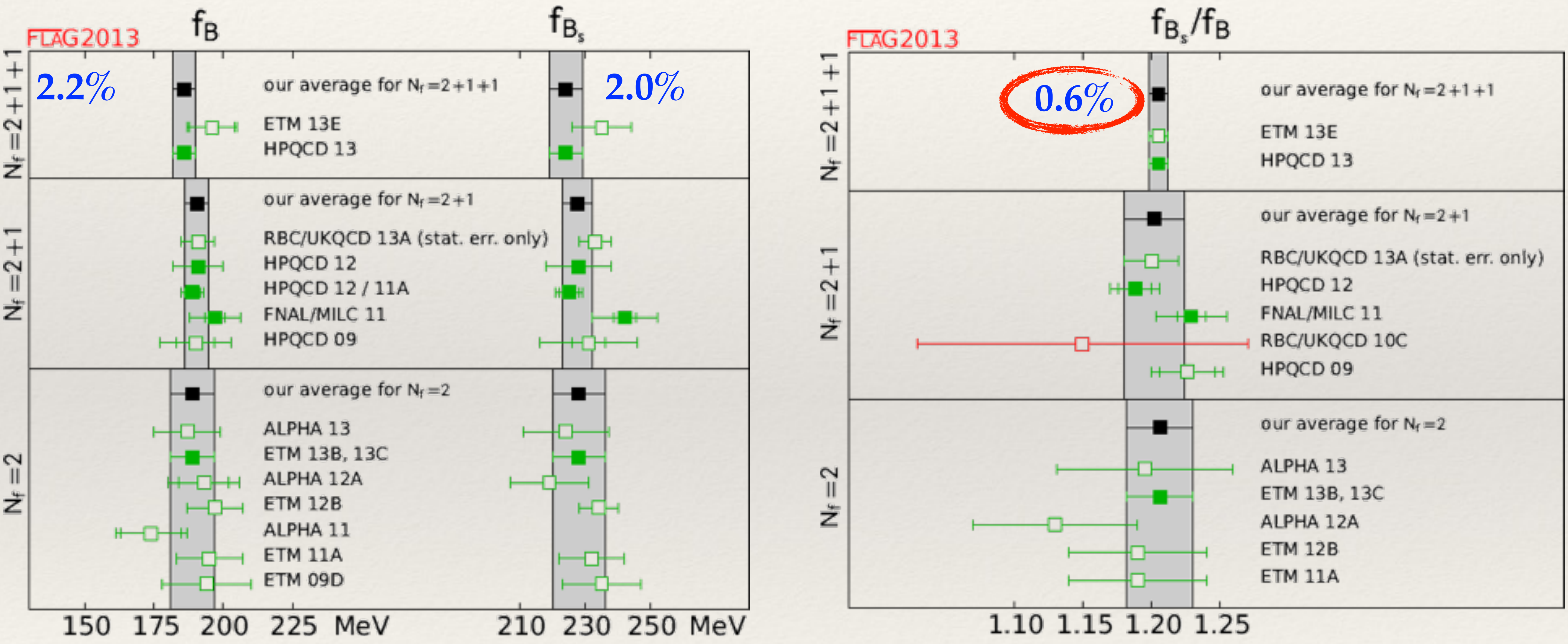
# Results for $|V_{cd}|$ and $|V_{cs}|$



- $|V_{cs}|$ from leptonic decays is slightly larger than from semileptonic decays
- $|V_{cs}|$ from leptonic decays is at tension with CKM-unitarity by 1.9σ (→HPQCD)



# Leptonic beauty decays



2.2%

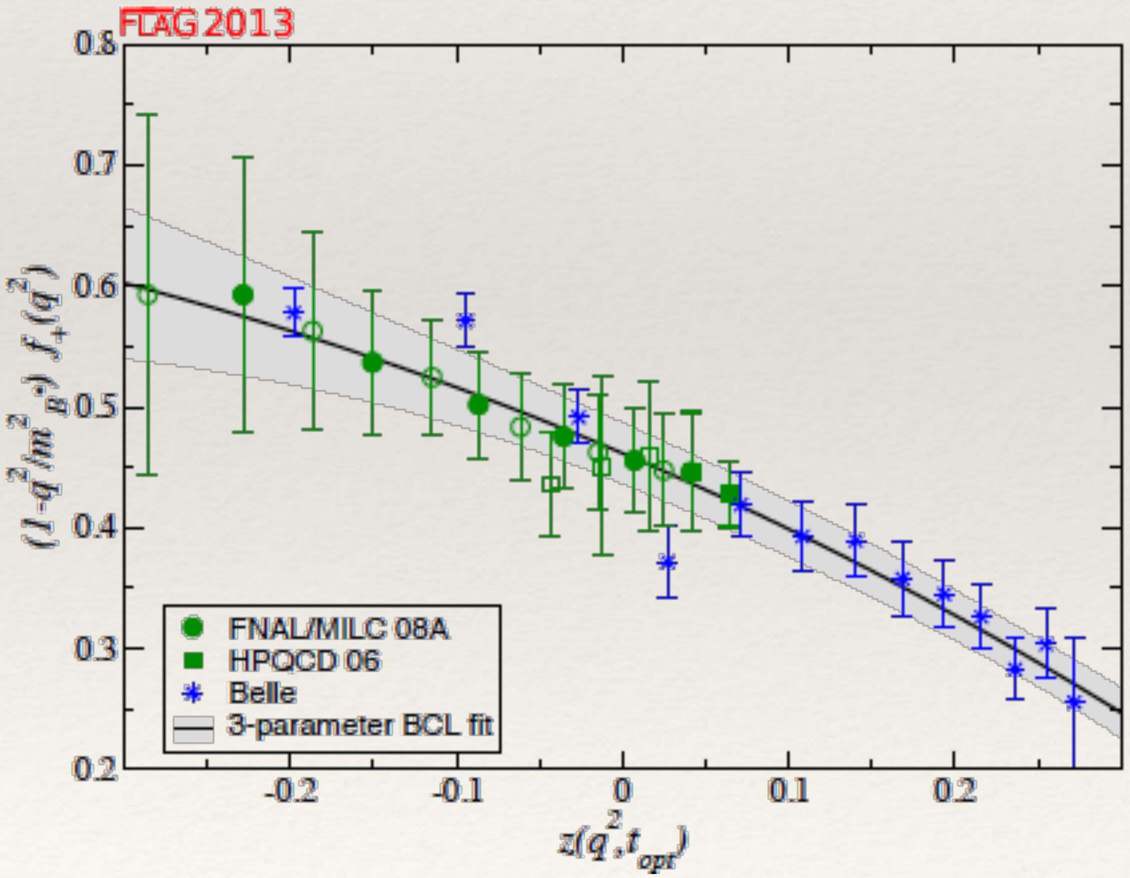
2.0%

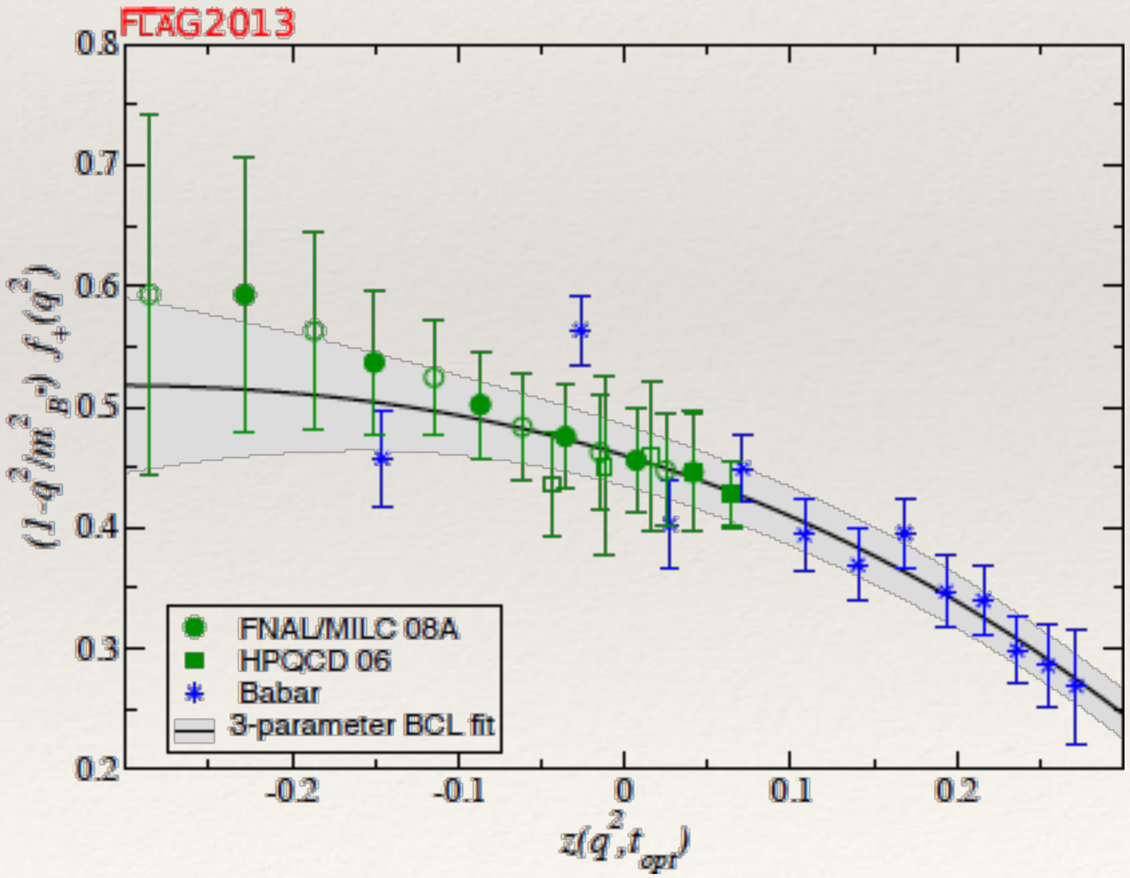
0.6%

# Semileptonic beauty decays

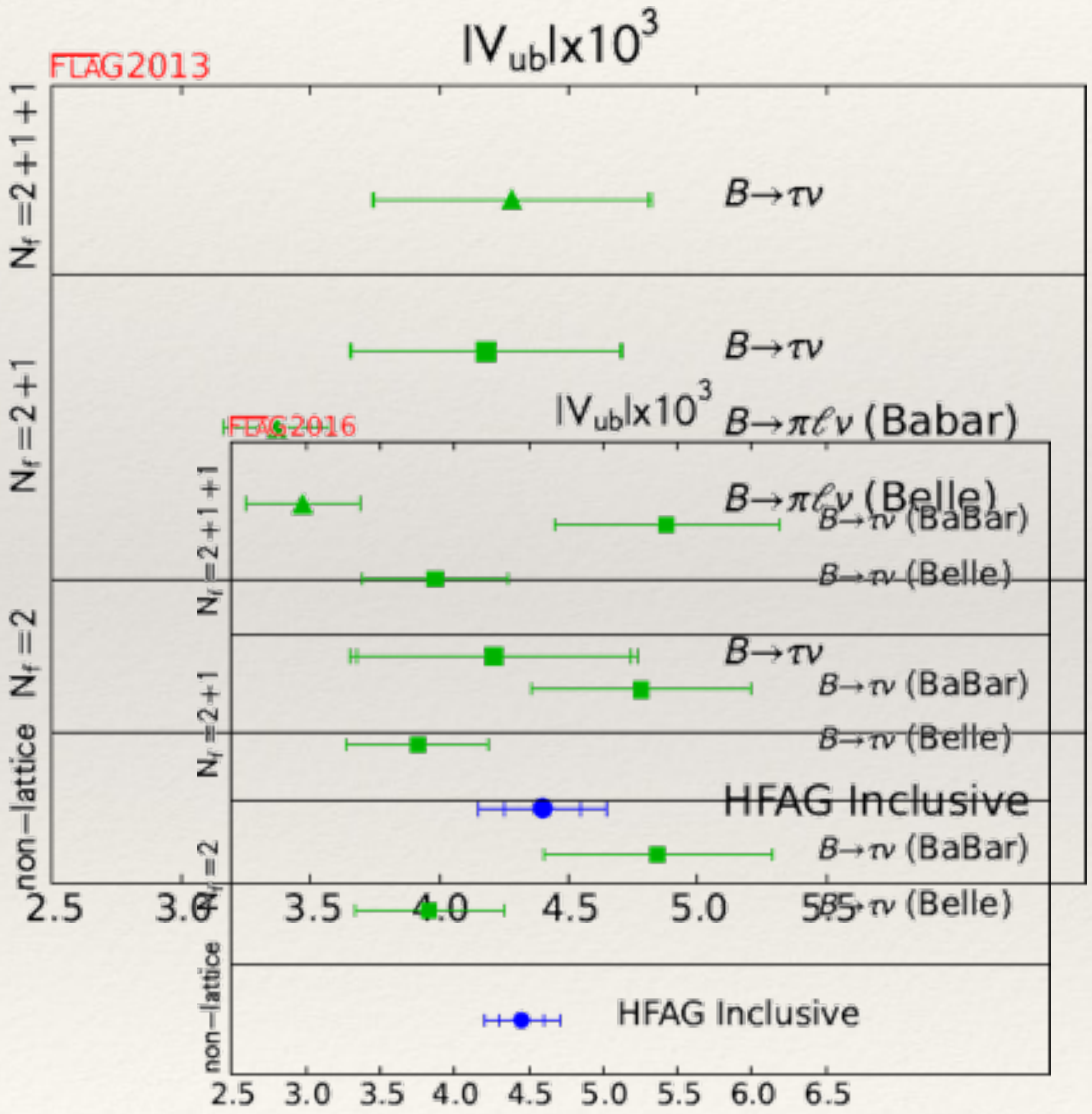
Kinematical reach limited in lattice QCD → extract value of $V_{ub}$ from simultaneous analysis of exp. and lattice data

$$q^2 = (E_B - E_{\text{light}})^2 - (\vec{p}_B - \vec{p}_{\text{light}})^2 \qquad \vec{p} = \frac{2\pi}{L}\vec{n}$$





# Results for $|V_{ub}|$



- tension between betw. incl. and excl. semilept. decays?
- lept. decay lies in between and agrees with both at 1.5σ

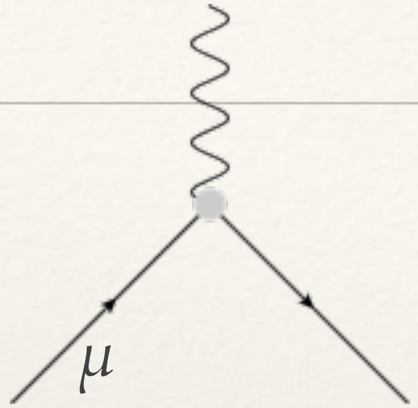
- slight tension in exp data

- looking forward to Belle II

# Flavour summary

- Lattice Flavour Physics is a mature research field
- many independent groups competing
- sub-percent precision for some quantities
- FLAG summarises particularly mature quantities for use in SM and BSM phenomenology

# $(g\text{-}2)_\mu$

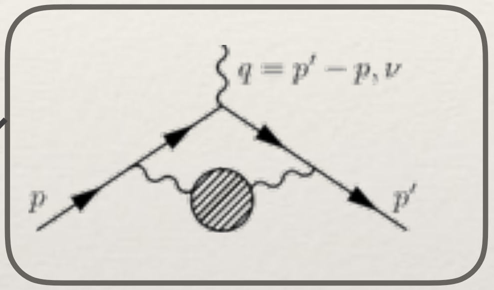


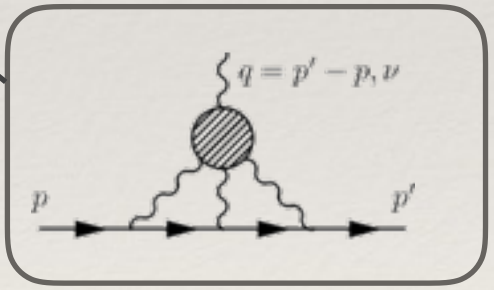
$$\langle e(\vec{p}\,')|j_\nu|e(\vec{p})\rangle = -e\,\bar{u}(\vec{p}\,')\left[F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right]u(\vec{p})$$

$$F_2(0) = \frac{g-2}{2} \equiv a_\mu$$

PDG 2013

| contribution | valx$10^{11}$ | δx$10^{11}$ |
|---|---:|---:|
| QED | 116584718.95 | 0.08 |
| EW | 153.6 | 1.0 |
| HVP LO | 6923 | 42 |
| HVP NLO | -98.4 | 0.6 |
| HVP LBL | 105 | 26 |
| SM | 116591803 | 49 |
| EXP | 116592091 | 63 |

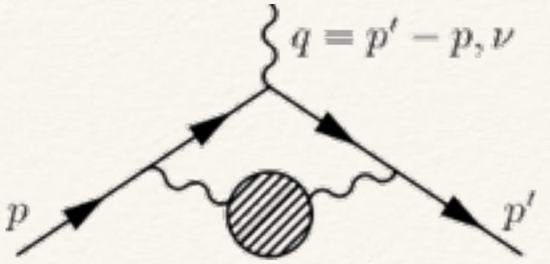




- discrepancy > 3σ
- new experiments (Fermilab, J-PARC)
- HVP LO from $e^+e^-$→hadrons
- HVP LBL model based

→ warrants any attempt at first principles computation

# LO HVP



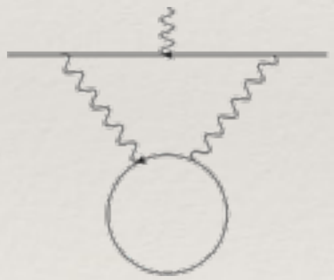
$$\boxed{a_\mu^{LO\,HVP} = 4\alpha^2 \int_0^\infty dQ^2 f(Q^2)\left(\Pi(Q^2) - \Pi(0)\right)}$$

Euclidean momenta!

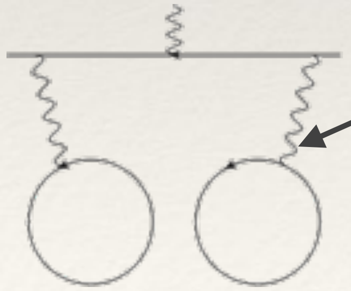
T. Blum PRL 91 (2003) 052001

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{-iQx} \underbrace{\boxed{\langle j_\mu(x) j_\nu(0)\rangle}}_{\text{vector-vector 2pt-function}} \qquad \Pi(Q^2) = \frac{\Pi_{\mu\nu}(Q^2)}{\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu}$$

connected

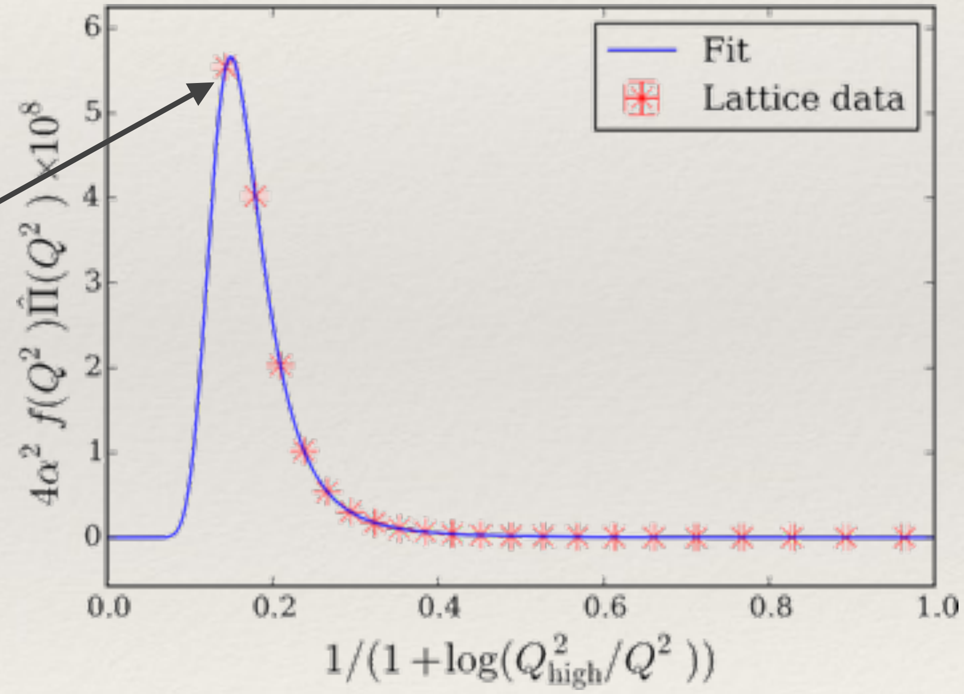


disconnected



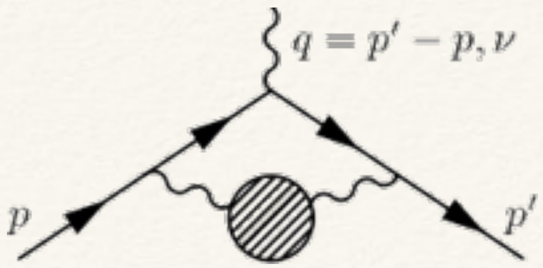
non-trivial:

- bad signal/noise ratio → stat. error
- integrand peaked at small $Q^2$ which is inaccessible on current lattices due to $p_i \sim 2\pi/L$
- $\Pi$ not defined at $Q^2 = 0$
- quark-disconnected contributions
- isospin breaking



But last years have seen tremendous progress !!!

# LO HVP



arXiv:1601.03071

JHEP 1604 (2016) 063 arXiv:1602.01767
arXiv:1512.09054

| $a_\mu$ x $10^{10}$ | HPQCD | RBC/UKQCD |
|---|---|---|
| light | 598(11) | work in progress |
| strange | 53.4(6) | 52.4(2.1) |
| charm | 14.4(4) | work in progress |
| disconnected | | −9.6(3.3)(2.3) |
| all | 666(6)(12) | — |
| SM OK exp all | 720(7) | 720(7) |

- strange, charm and bottom sufficiently precisely known
- getting the disconnected in full LQCD was a big achievement (previously considered show stopper)

- first results (HPQCD) indicate tension confirmed

Need to concentrate on:

- stat. error on light contribution
- strong and elm. isospin breaking effects (later)

There is significant work on light-by-light going on as well!

Blum et al., Phys.Rev. D93 (2016) no.1, 014503 arXiv:1510.07100
Green et al., Phys.Rev.Lett. 115 (2015) no.22, 222003 arXiv:1507.01577

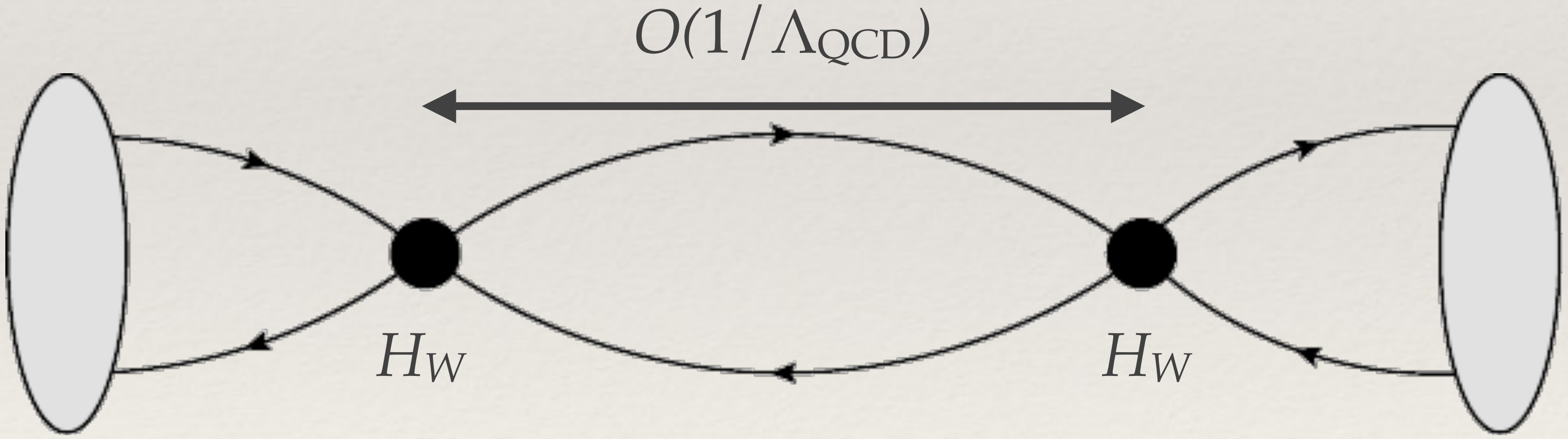
# Beyond precision lattice QCD

Go beyond factorisation

$$\Gamma_{\text{exp.}} \overset{???}{=} V_{\text{CKM}}(\text{WEAK})(\text{EM})(\text{STRONG})$$

treat jointly in lattice QCD+QED

Go beyond short distance physics



# Including QED in meson decay MEs

- Precision on MEs such that EM and strong isospin effects important remember: so far mostly only QCD ($m_l = m_u = m_d$, $\alpha_{EM} = 0$)
- we should go beyond EFT treatment (e.g. replace ChPT estimates)
- need to understand how this can be done conceptually
- already many results for spectroscopic quantities but not for matrix elements
- finite size effects with photons pose a substantial problem

# QCD+QED

Action:

$$S[U,A,\bar{\psi},\psi] = S_g[U;g] + S_\gamma[A] + \sum_f \bar{\psi}_f D[U,A;e,q_f,m_f]\psi_f$$

$$S_\gamma^{naive} = -\frac{a^4}{4}\sum_{\mu,\nu,x}(\partial_\mu A_{\nu,x} - \partial_\nu A_{\mu,x})^2$$

- MC simulation of discretised theory
- QCD has a mass gap → finite volume effects $\propto e^{-m_\pi L}$ (for simple MEs)
- photon is massless and interacts over long range
  → power-like finite volume effects $\propto 1/L, 1/L^2, \ldots$ from exchange of photon around torus
- sufficiently large volumes currently not feasible, so use effective field theory to subtract finite volume effects

# QCD+QED

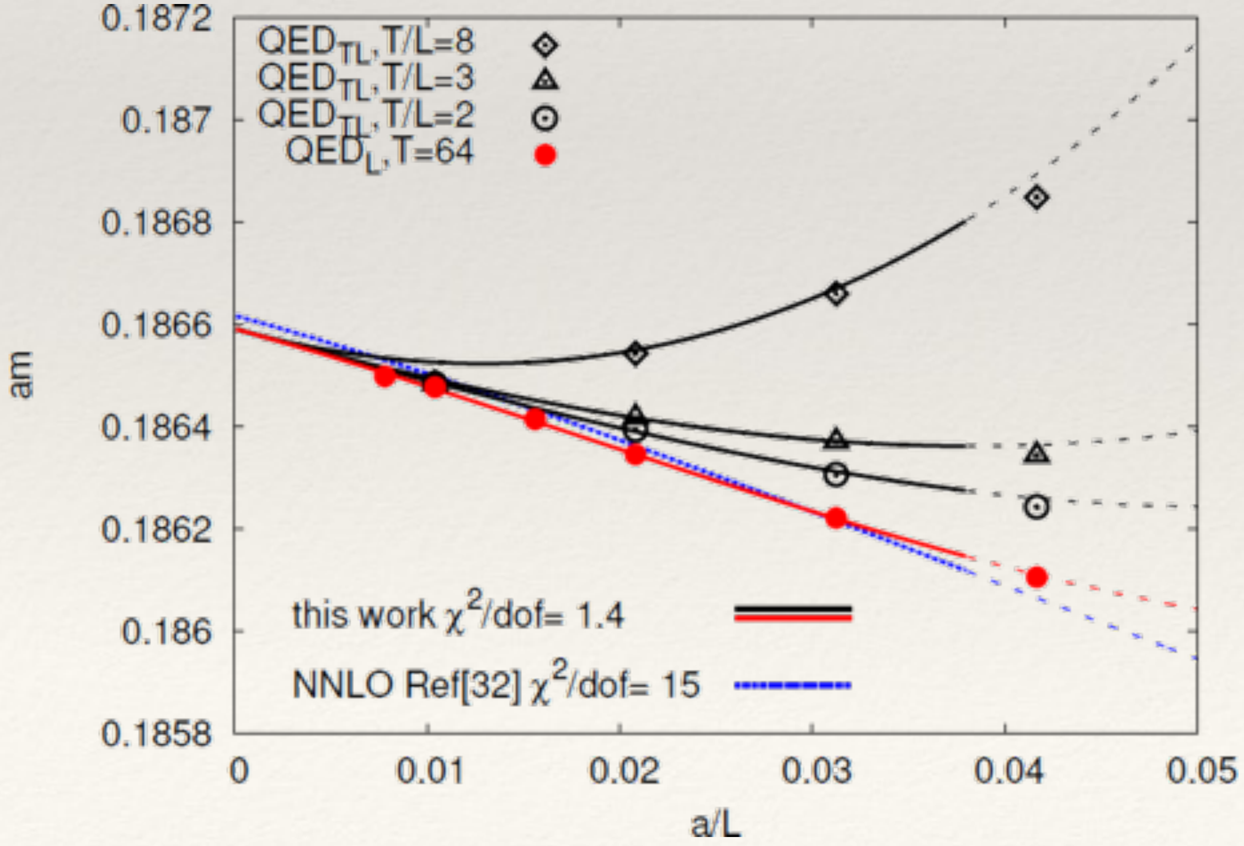
**BMW Collaboration**
**Science 347 (2015) 1452-1455**
**arXiv:1406.4088**

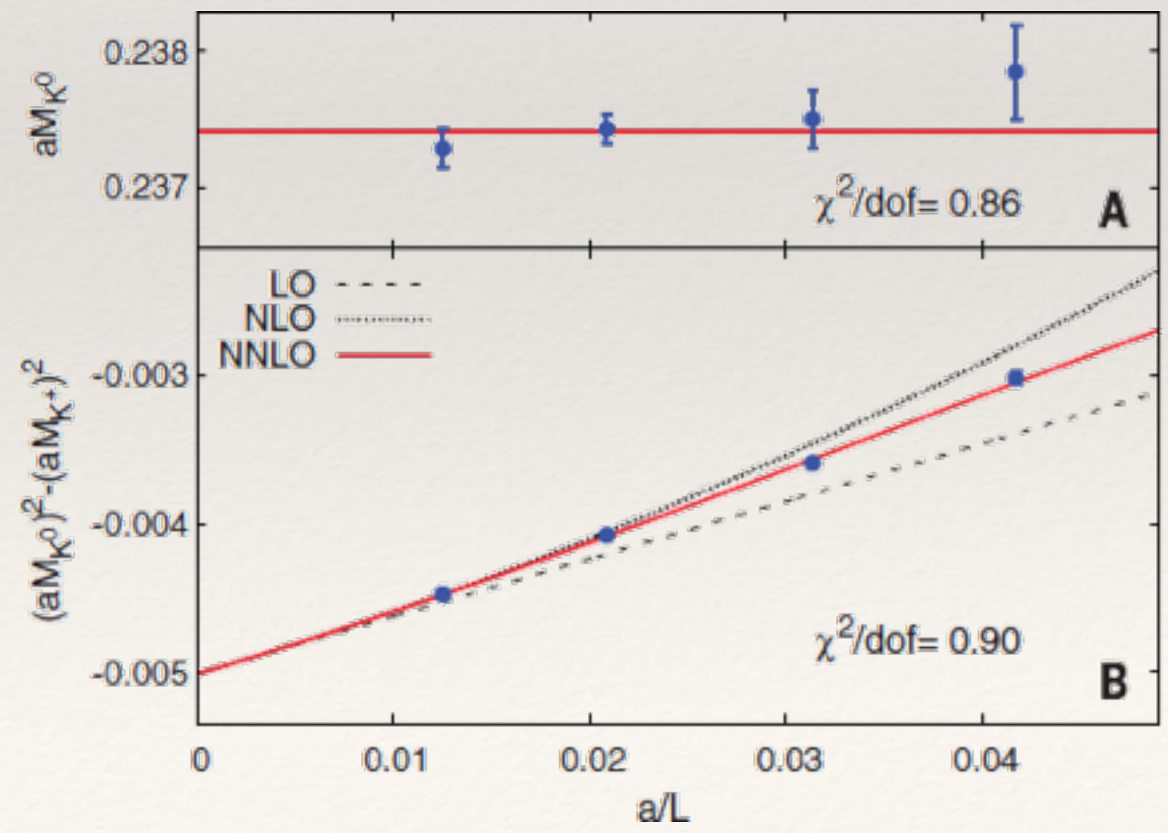
Example: FV correction to mass of a spin-1/2 particle in QED

analytically compute the difference of the *finite volume* and *infinite volume* self energies Σ:

$$m^2(T,L) \overset{T,L\to\infty}{\propto} m^2\left\{1 - q^2\alpha\left[\frac{\kappa}{2mL}\left(1+\frac{2}{mL}\right) - \frac{3\pi}{(mL)^3}\right]\right\}$$

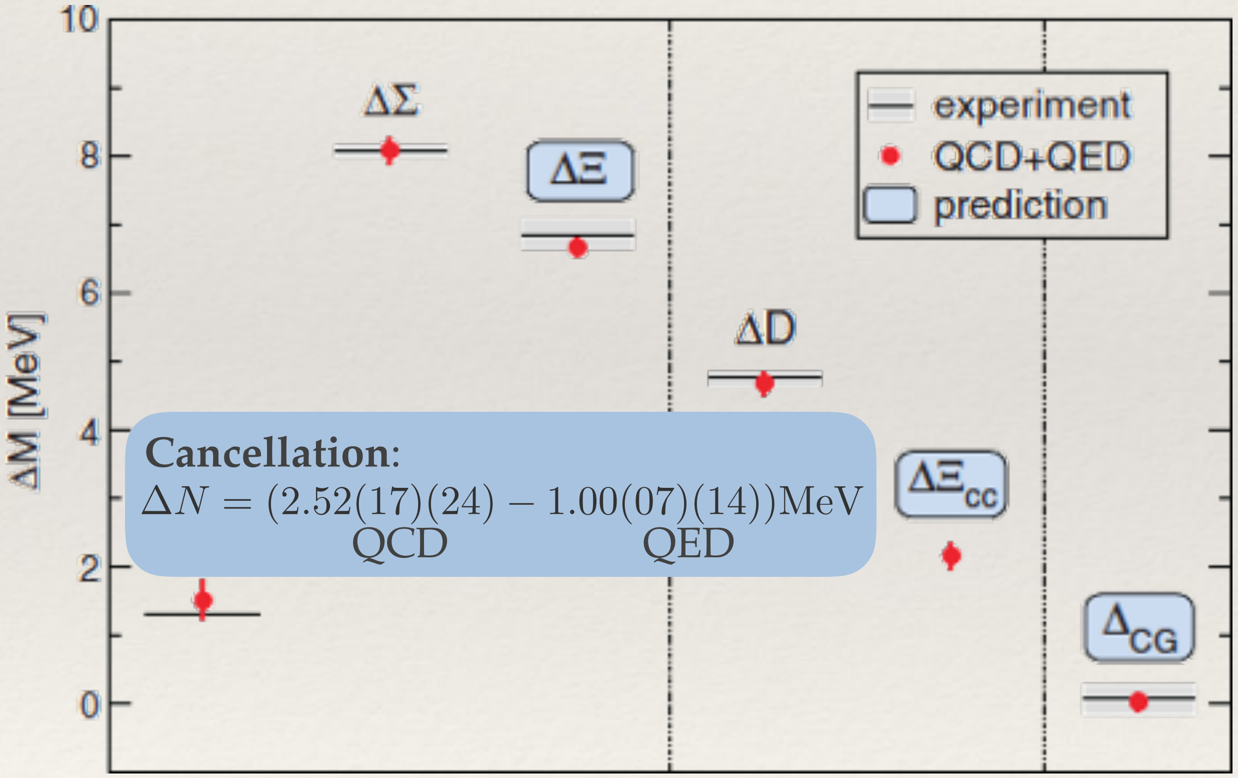
leading behaviour universal in *κ* (structure- and spin-independent)





# QCD+QED: baryon mass splitting

- relative neutron-proton mass difference found in nature 0.14%
- the value has significant implication for nature
  - smaller value → inverse $\beta$-decay of $H$
  - much larger value → faster $\beta$-decay for neutrons in BBN



**Cancellation:**

$$\Delta N = (\underset{\text{QCD}}{2.52(17)(24)} - \underset{\text{QED}}{1.00(07)(14)})\text{MeV}$$

BMW carried out simulations of $N_f = 1{+}1{+}1{+}1$ *QCD+QED* simulations and determined the light baryon isospin splitting

**BMW Collaboration**
**Science 347 (2015) 1452-1455**
**arXiv:1406.4088**

# Including QED in meson decay MEs

- leptonic decay at $O(\alpha^0)$:

$$\Gamma(\pi^+ \to l^+ \nu_l) = \frac{G_F^2 |V_{ud}|^2 f_\pi^2}{8\pi} m_\pi m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2$$

- including elm. effects @ $O(\alpha)$:

$$\begin{aligned} \Gamma(\pi^+ \to l^+ \nu_l(\gamma)) &= \Gamma(\pi^+ \to l^+ \nu_l) + \Gamma(\pi^+ \to l^+ \nu_l \gamma) \\ &\equiv \qquad \Gamma_0 \qquad\qquad + \qquad \Gamma_1 \end{aligned}$$

IR div. cancel between terms on r.h.s.
between virtual and real photons
(Bloch Nordsieck)

# Including QED in meson decay MEs

Carrasco et al. PRD 91 074506 (2015) **arXiv:1502.00257**

- cut on small photon momentum $< \Delta E \rightarrow \gamma$ *sees* point-like $\pi$
  $\Delta E \approx 20$MeV experimentally accessible and $\pi$ point like

point approximation

$$\Gamma(\Delta E) = \lim_{V\to\infty} \boxed{(\Gamma_0 - \Gamma_0^{\mathrm{pt}})} + \lim_{V\to\infty} \boxed{(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E))}$$

$\Gamma(\pi^+ \rightarrow l^+ \nu_l)$

lattice and analytical finite $V$

$\Gamma(\pi^+ \rightarrow l^+ \nu_l \gamma(\Delta E))$

analytically in $V \rightarrow \infty$

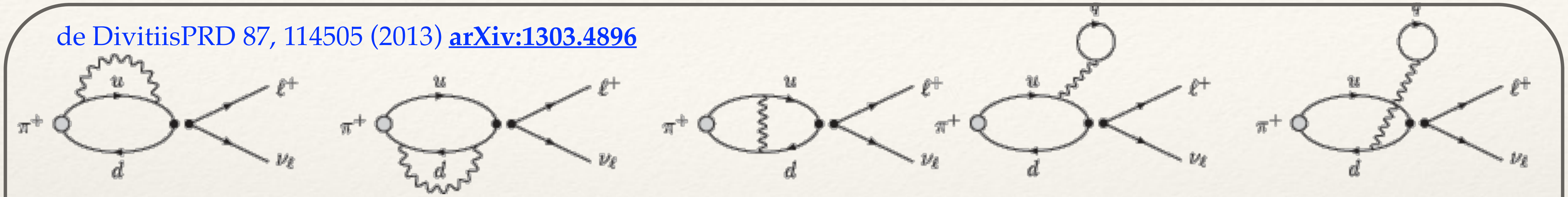
both terms separately IR finite, gauge invariant on its own

- analytical calculation for pt. approximation is done:

$$\mathcal{L}_{\pi\text{-}\ell\text{-}\nu_\ell} = i\, G_F f_\pi V_{ud}^* \; \{(\partial_\mu - ieA_\mu)\pi\} \left\{ \bar{\psi}_{\nu_\ell} \frac{1+\gamma_5}{2} \gamma^\mu \psi_\ell \right\} + \mathrm{H.C.}\,.$$

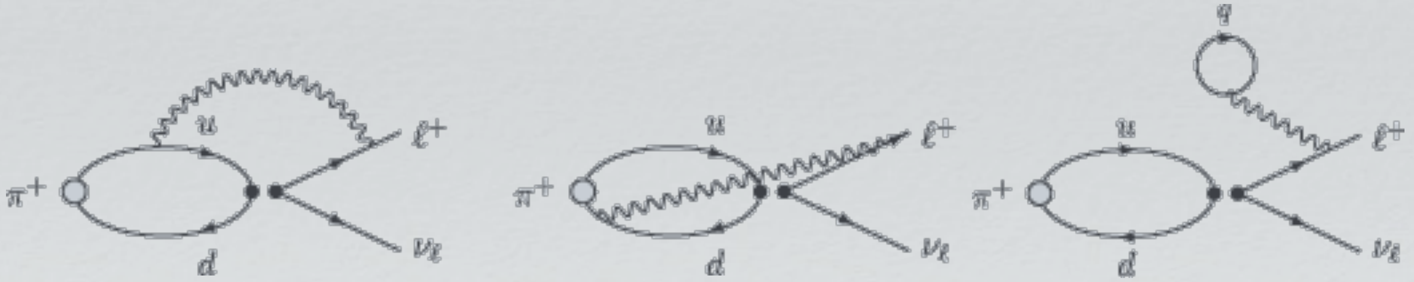
# Including QED in meson decay MEs

de DivitiisPRD 87, 114505 (2013) **arXiv:1303.4896**



$$C_1(t) = -\frac{1}{2}\int d^3\vec{x} d^4x_1 d^4x_2 \sum_{\vec{x}} \langle 0|T\left\{J_W^\nu(0) j_\mu(x_1) j_\mu(x_2)\phi^\dagger(\vec{x},-t)\right\}|0\rangle \Delta(x_1,x_2)$$

$$C_0(t) + C_1(t) \simeq \frac{Z^\phi}{2m_\pi} e^{-m_\pi t}\mathcal{A} \simeq \frac{Z_0^\phi + \delta Z^\phi}{2(m_\pi^0 + \delta m_\pi)} e^{-m_\pi^0 t}\left(1 - \delta m_\pi t\right)\left(\mathcal{A}_0 + \delta\mathcal{A}\right)$$

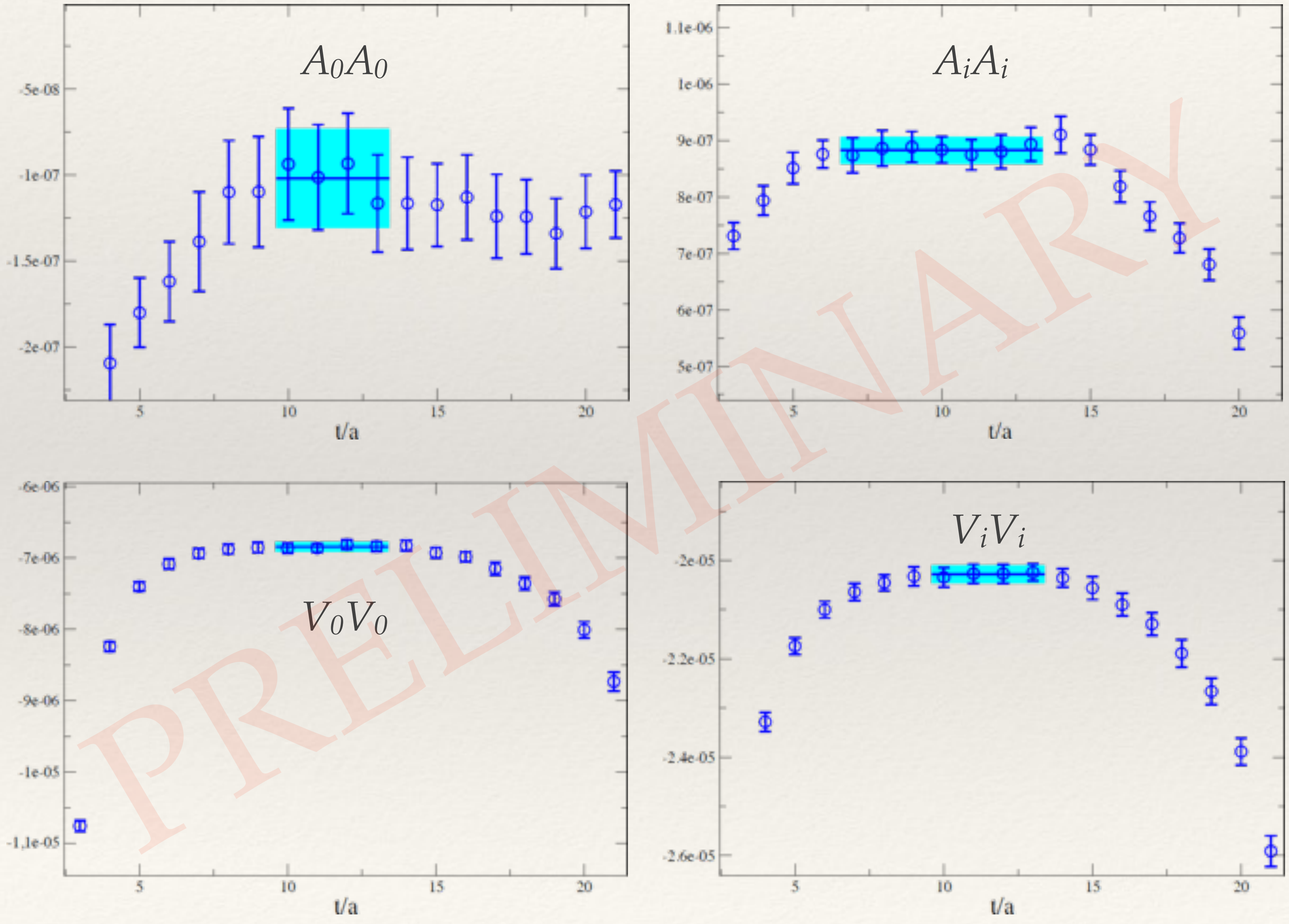


$$C_1(t)_{\alpha\beta} = -\int d^3\vec{x} d^4x_1 d^4x_2 \langle 0|T\left\{J_W^\nu(0) j_\mu(x_1)\phi^\dagger(\vec{x},-t)\right\}|0\rangle$$
$$\times\Delta(x_1,x_2)\left(\gamma_n u(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta} e^{E_l t_2 - i\vec{p}_l\cdot\vec{x}_2}$$

convergent by
energy / momentum conservation

# Including QED in meson decay MEs

$24^{3;}\ m_\pi \approx 500\text{MeV}$



- first time ever conceptually clean attempt of calculation of leptonic decay at $O(\alpha)$
- disconnected pieces need to be included
- $\Gamma_0$ works, now needs to be combined wt. analytical results for $\Gamma_0^{\rm pt}$, $\Gamma_1^{\rm pt}(\Delta E)$
- ~20% stat. error would be sufficient for use in phenomenology

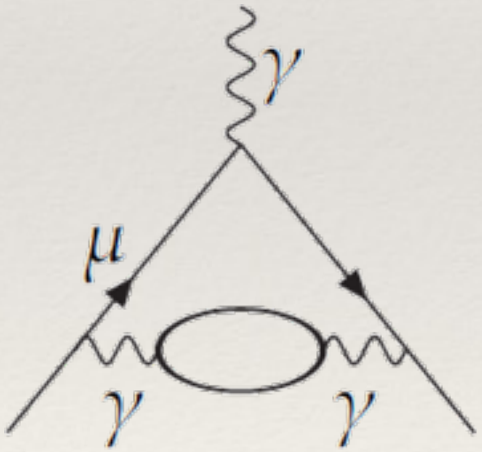
# QCD+QED applications

- start with light flavour matrix elements $f_\pi$, $f_K$, $f_+(0)$, …

$$\Gamma_{K\to\pi l\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^2} S_{\mathrm{EW}} (1 + \Delta_{SU(2)} + \Delta_{\mathrm{EM}})^2 \, I \, |f_+^{K\pi}(0)|^2 |V_{us}|^2$$

$$\langle \pi(p_\pi)|V_\mu(0)|K(p_K)\rangle = f_+^{K\pi}(q^2)(p_K + p_\pi)_\mu + f_-^{K\pi}(q^2)(p_K - p_\pi)_\mu$$

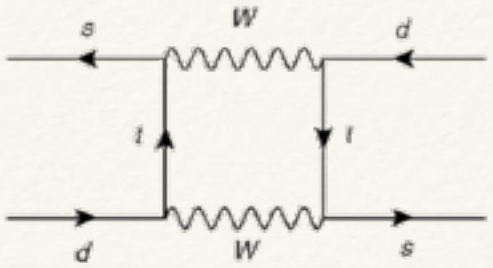
- lattice predictions of leading hadronic contribution to muon g-2

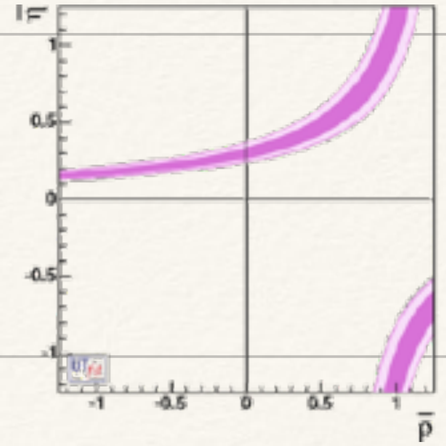


  - lattice (isospin symmetric, $\alpha_{\mathrm{EM}}=0$ is getting competitive with experimental determination *($e^+e^- \to$ hadrons)*
  - next step would be inclusion of isospin breaking effects

- inclusion of QED effects will be one of the big challenges over the next years

# Long distance effects in kaon physics - mixing



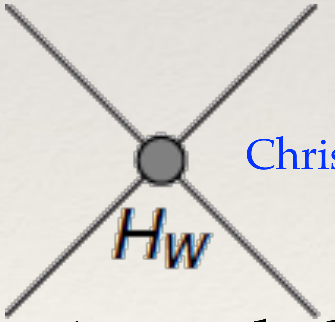


$$\epsilon_K = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})} = e^{i\Phi_\epsilon} \sin\phi_\epsilon \left( \frac{\mathrm{Im}\langle \bar{K}^0 | H_W^{\Delta S=2} | K^0 \rangle}{\Delta M_K} + \text{L.D. effects} \right)$$

Buras, Guadagnoli PRD 78 (2008)
Buras, Guadagnoli, Isidori,
PLB 688 (2010)

Long Distance effects amount to O(5%), so certainly worth considering on the lattice

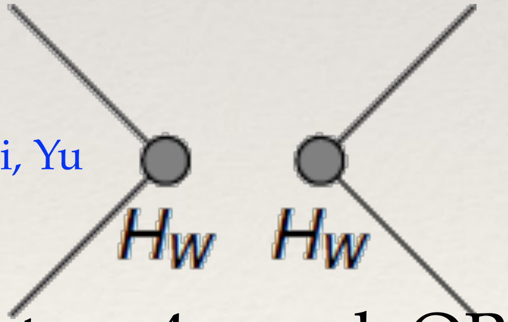
1st order Weak



single 4-quark OP,
length scale $10^{-18}$m

Christ, Izubuchi, Sachrajda, Soni, Yu
arXiv:1212.5931

2nd order Weak

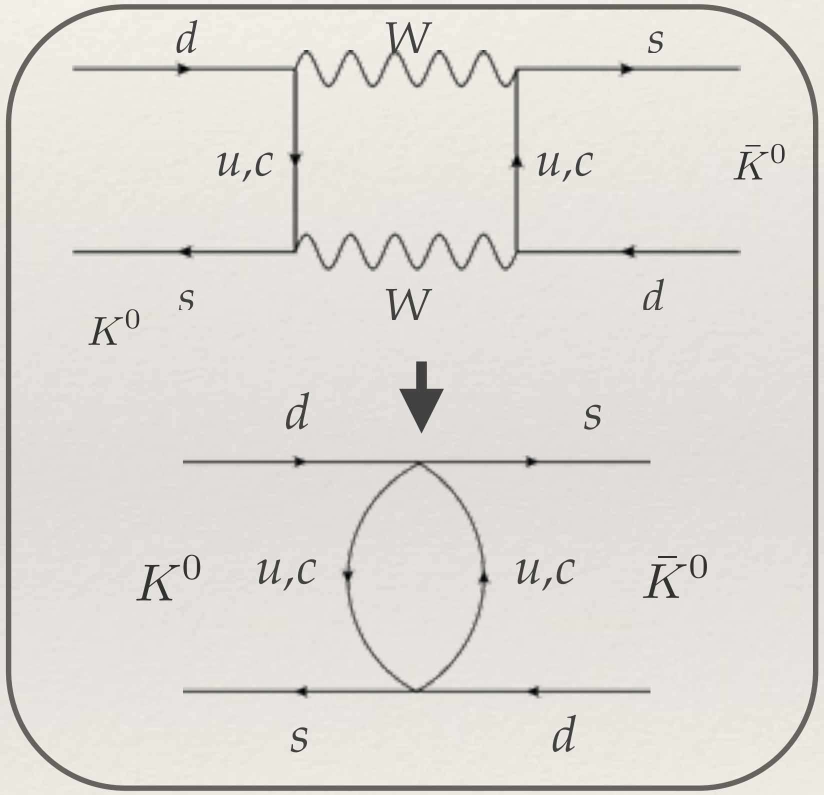


two 4-quark OP
length scale $1/\Lambda_{QCD}$

# Beyond short distance: e.g. $\Delta M_K$

$$\boxed{\Delta M_K = m_{K_S} - m_{K_L} = \boxed{2\mathrm{Re}M_{\bar{0}0}}} \qquad M_{\bar{0}0} = \mathcal{P}\sum_\lambda \frac{\langle \overline{K^0}|H_W|\lambda\rangle\langle\lambda|H_W|K^0\rangle}{m_K - E_\lambda}$$

- experimentally $\Delta M_K$=3.483(6)×$10^{-12}$MeV (PDG)
- suppressed by 14 orders of magnitude with respect to QCD → poses strong BSM constraints (e.g. $(1/\Lambda)^2$ $\bar{s}d\bar{s}d$ BSM contribution) knowing $\Delta M_K$ at 10%-level → **$\Lambda \geq 10^4$TeV**
- SD about 70% of experimental value - rest LD?
- PT large contributions at $\mu \sim m_c$ where PT turns out to converge badly (NLO->NNLO constitutes 36% correction) Brod, Gorbahn PRL 108 121801 (2012) arXiv:1108.2036



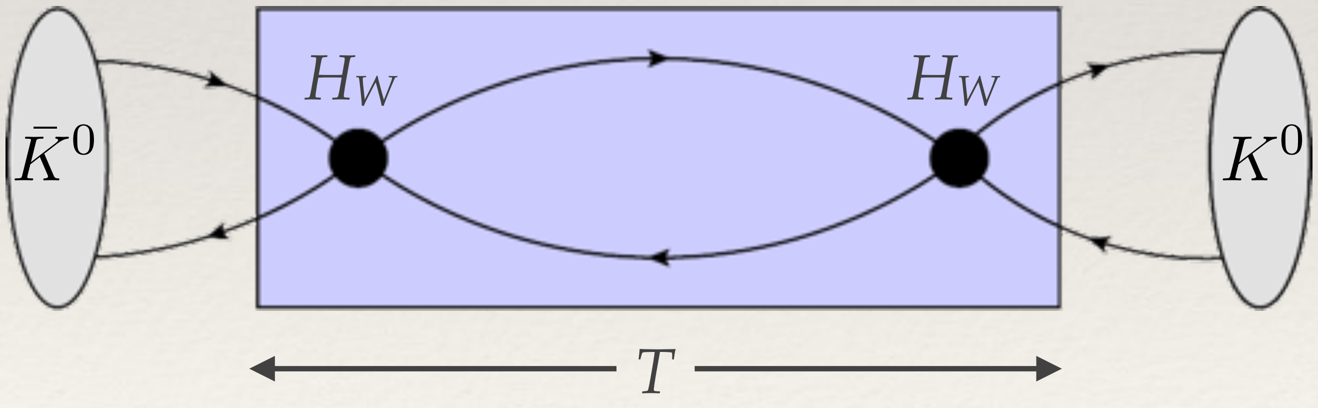
# long distance effects – $\Delta M_K$

N. Christ et al. PRD 88 (2013) 014508 **arXiv:1212.5931**
Bai et al. PRL 113 (2014) 112003 **arXiv:1406.0916**

$$M_{\bar{0}0} = \mathcal{P}\sum_{\lambda} \frac{\langle \overline{K^0}|H_W|\lambda\rangle\langle\lambda|H_W|K^0\rangle}{m_K - E_\lambda}$$

$$\mathcal{A} = \langle 0|T\left\{K^0(t_f)\frac{1}{2}\int_{t_A}^{t_B} dt_2 \int_{t_A}^{t_B} dt_1 H_W(t_2)H_W(t_1){K^0}^\dagger(t_i)\right\}|0\rangle$$

Integrate operators (here $H_W$) over time interval where initial and final kaon dominate



# long distance effects – $\Delta M_K$

N. Christ et al. PRD 88 (2013) 014508 **arXiv:1212.5931**
Bai et al. PRL 113 (2014) 112003 **arXiv:1406.0916**

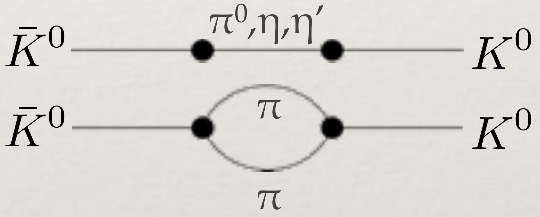
$$\mathcal{A} = N_K^2 e^{-M_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{M_K - E_n} \left( -T - \frac{1}{M_K - E_n} + \frac{e^{(M_K - E_n)T}}{M_K - E_n} \right)$$

amplitude
$\Delta m_K^{\mathrm{FV}}$

irrelevant
constant

exponential term
needs to be subtracted

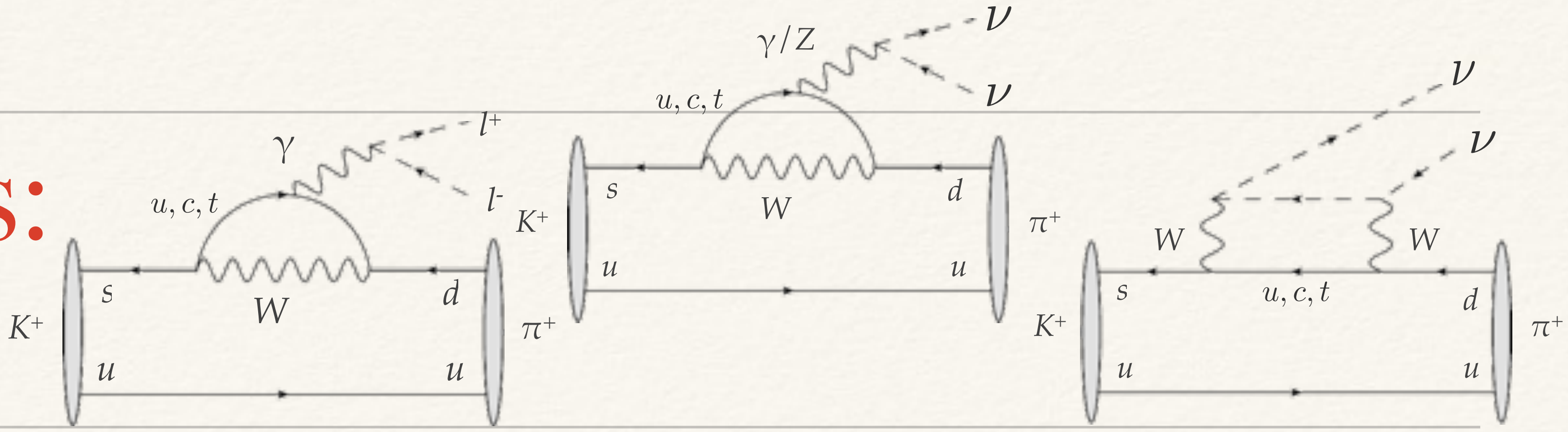
- multiple hadrons in intermediate states causing difficulties and need to be subtracted



- finite volume corrections from two-particle intermediate state can be sizeable
  N. Christ et al. PRD91 (2015) 114510 **arXiv:1504.01170** also: Briceno, Hansen **arXiv:1502.04314**
  extension of Lellouch-Lüscher correction to 2nd order weak MEs
  $\Delta^{\mathrm{FV}}\left(\Delta M_K\right) = -\cot\left(\phi(M_K) + \delta_0(M_K)\right) \frac{d(\phi(E)+\delta_0(E))}{dE}\Big|_{E=M_K} |\langle \bar{K}^0 | H_W | \pi\pi, M_K \rangle^{\mathrm{V}'}|^2$

- what happens when the two $H_W$ approach each other (GIM in action)?

# long distance effects: Rare kaon decays



Two new experiments dedicated to rare kaon decays NA62 (CERN) and KOTO (J-PARC) are running

- FCNC (W-W or γ/Z-exchange diagrams)
- deep probe into flavour mixing and SM/BSM due to suppression in the SM
- can determine $V_{td}$, $V_{ts}$ and test SM

$K_L \to \pi^0 \nu\bar{\nu}$

- KOTO (J-PARC)
- direct CP violation
- exp. BR $\leq 2.6 \times 10^{-8}$
  theory BR $3.0(3) \times 10^{-11}$
- GIM → top dominated and charm suppressed, purely SD

$K^+ \to \pi^+ \nu\bar{\nu}$

- NA62 (CERN)
- CP conserving
- exp. BR $1.73(^{+1.15}_{-1.05}) \times 10^{-10}$
  theory BR $0.911(72) \times 10^{-10}$
- small LD contribution, candidate for lattice

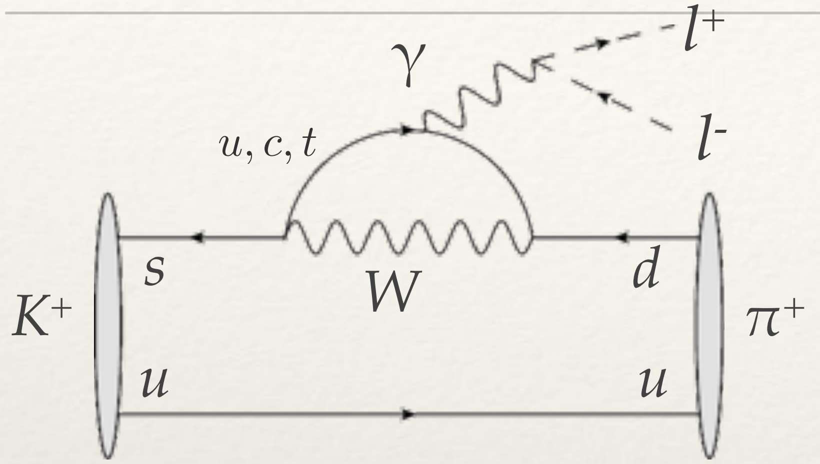
compute in lattice QCD

$K^+ \to \pi^+ l^+ l^-$ $K_S \to \pi^0 l^+ l^-$

- 1-photon exchange LD dominated
- indirect contribution to CP-violating rare $K_L$ decay
- SM prediction mainly ChPT
- lattice can predict ME and LECs
- experimenters will be able to look at these channels as well

# Rare kaon decays $K^+ \to \pi^+ l^+ l^-$

**N. Christ et al.** arXiv:1507.03094
arXiv:1602.01374



LD contribution given through $K \to \gamma^*$ contribution which is computed as

$$\mathcal{A}_\mu = (q^2) \int d^4x \langle \pi(p) | T\left[ J_\mu(0) H_W(x) \right] | K(k) \rangle$$

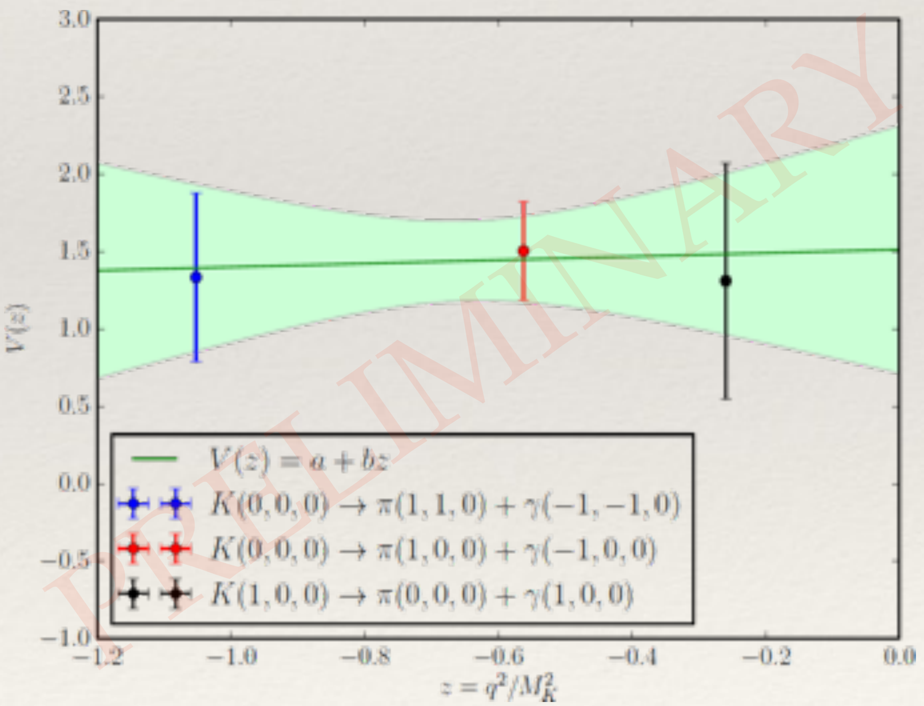
dominant operators: $Q_1^q = \left(\bar{s}_i \gamma_\mu^L d_i\right)\left(\bar{q}_j \gamma_\mu^L q_j\right)$, $\quad Q_2^q = \left(\bar{s}_i \gamma_\mu^L q_i\right)\left(\bar{q}_j \gamma_\mu^L d_j\right)$

Decay amplitude in terms of elm. transition form factor

$$A_i = -\frac{G_F \alpha}{4\pi} V_i(z) (k+p)^\mu \bar{u}_l(p_-) \gamma_\mu \nu_l(p_+) \qquad (i = +, S)$$

$$V_i(z) = a_i + b_i z + V_i^{\pi\pi}(z)$$

- the $a_S$ and $a_+$ can be extracted from experiment or lattice
- $a_S$ parameterises also the CP-violating contribution to the $K_L$ decay



# Summary/Conclusions

- considerable set of SM parameters, spectra and matrix elements now reliably and precisely predicted in full lattice QCD
- Flavour Lattice Averaging Group (FLAG)
- precision such that isospin breaking effects in matrix elements and spectra needs to be taken into account
- long distance effects:
  - neutral kaon system
  - rare kaon decays (new experimental facilities!!!)

  with loads of new questions and theoretical problems and potential impact on SM and BSM phenomenology

# Summary/Conclusions

- this talk is by far not inclusive:
  - $K \to \pi\pi$, $g$-2, …
  - finite-$T$,$\mu$
  - BSM
  - …

34th International Symposium on
Lattice Field Theory

University of Southampton
24–30 July 2016

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