Liverpool, 19-11-2014

# The three-loop splitting functions in QCD: the helicity-dependent case

**Andreas Vogt (University of Liverpool)**

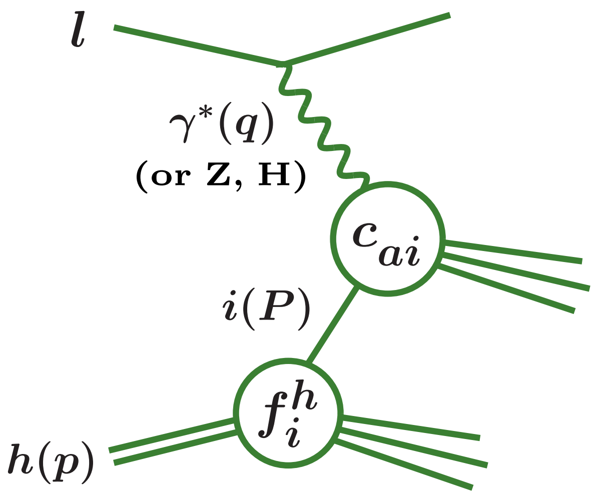
**with Sven Moch (Hamburg Univ.) and Jos Vermaseren (NIKHEF)**

- Polarized PDFs, their evolution, $\alpha_s^2$ calculations (1990s), large-$x$ limit
- $\alpha_s^3$ via $g_1^{\mathrm{e.m.}}$ (2008, all-$N$) & graviton-exch. DIS (new, extreme Mincer)
- All-$N$ expressions, via end-point knowledge and number theory tools

arXiv: 0807.1238 (LL ’08); arXiv: 1405.3407 (LL ’14), arXiv: 1409.5131 (Nucl. Phys. B)

# Hard lepton-hadron processes in pQCD (I)

**Inclusive deep-inelastic scattering (DIS), semi-incl. $l^+l^-$ annihilation (SIA)**




**Left $\longrightarrow$ right: DIS, $q$ spacelike, $Q^2 = -q^2$**

$P = \xi p$, $f_i^h =$ **parton distributions**

**Top $\longrightarrow$ bottom: $l^+l^-$, $q$ timelike, $Q^2 = q^2$**

$p = \xi P$, **fragmentation distributions**

**Drell-Yan $l^+l^-$, Higgs prod'n: bottom $\longrightarrow$ top, 2nd hadron from below ($\{\ldots\}$)**

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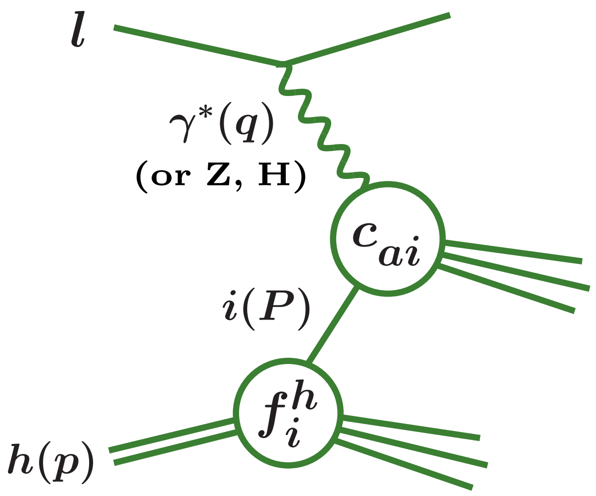


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**Specific for observables $F_a$ (structure functions etc): coefficient functions**

$$F_a(x, Q^2) \;=\; \Big[ C_{a,i\{j\}}(\alpha_{\mathrm{S}}(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\} \Big](x) + \mathcal{O}(1/Q^{(2)})$$

**Scaling variables: $x = Q^2/(2p\!\cdot\! q)$ in DIS etc. $\mu$: renorm./mass-fact. scale**

# Hard lepton-hadron processes in pQCD (II)

**Universal parton / fragmentation distributions $f_i$ : evolution equations**

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) = \left[P_{ik/ki}^{(S)/T}(\alpha_S(\mu^2)) \otimes f_k(\mu^2)\right](\xi) \ , \quad \otimes : \text{ Mellin convolution}$$

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**Expansion in $a_s = \alpha_s(\mu^2)/4\pi$: splitting functions $P$, coefficient fct's $C_a$**

$$\begin{aligned} P \;&=\; a_s\, P^{(0)} + a_s^2\, P^{(1)} + a_s^3\, P^{(2)} + a_s^4\, P^{(3)} + \ldots \\ C_a \;&=\; \underbrace{a_s^{n_a}\Big[\, c_a^{(0)} + a_s\, c_a^{(1)}}\; + a_s^2\, c_a^{(2)} \;+\; a_s^3\, c_a^{(3)} \;+\; \ldots \Big] \end{aligned}$$

**NLO: first real prediction of size of cross sections**

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NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate

N$^3$LO: for high precision ($\alpha_{\mathsf{S}}$ from DIS), slow convergence (Higgs in $pp/p\bar{p}$ )

$F_2/F_3$ in DIS: MVV (2005/8); $\sigma_{\mathrm{H,soft+virtual}}$ : Anastasiou et al (2014)

Endpoint logs for $x \rightarrow 0, 1$: resummation can be useful or necessary

# Polarized (singlet) PDFs and their evolution

Long. polarized proton: q/g distributions $f_i^{\rightarrow}, f_i^{\leftarrow}$ for same, opposite helicity

Unpolarized and polarized parton distribution functions (PDFs)

$$\begin{aligned} f_i(x,\mu^2) &= f_i^{\rightarrow}(x,\mu^2) + f_i^{\leftarrow}(x,\mu^2) \\ \Delta f_i(x,\mu^2) &= f_i^{\rightarrow}(x,\mu^2) - f_i^{\leftarrow}(x,\mu^2) \end{aligned}$$

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**Symmetries: $2n_f - 1$ scalar evolution equations and $2 \times 2$ system**

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} \Delta f_{\mathrm{q}} \\ \Delta f_{\mathrm{g}} \end{pmatrix} = \begin{pmatrix} \Delta P_{\mathrm{qq}} & \Delta P_{\mathrm{qg}} \\ \Delta P_{\mathrm{gq}} & \Delta P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} \Delta f_{\mathrm{q}} \\ \Delta f_{\mathrm{g}} \end{pmatrix} \equiv \Delta P \otimes \Delta f$$

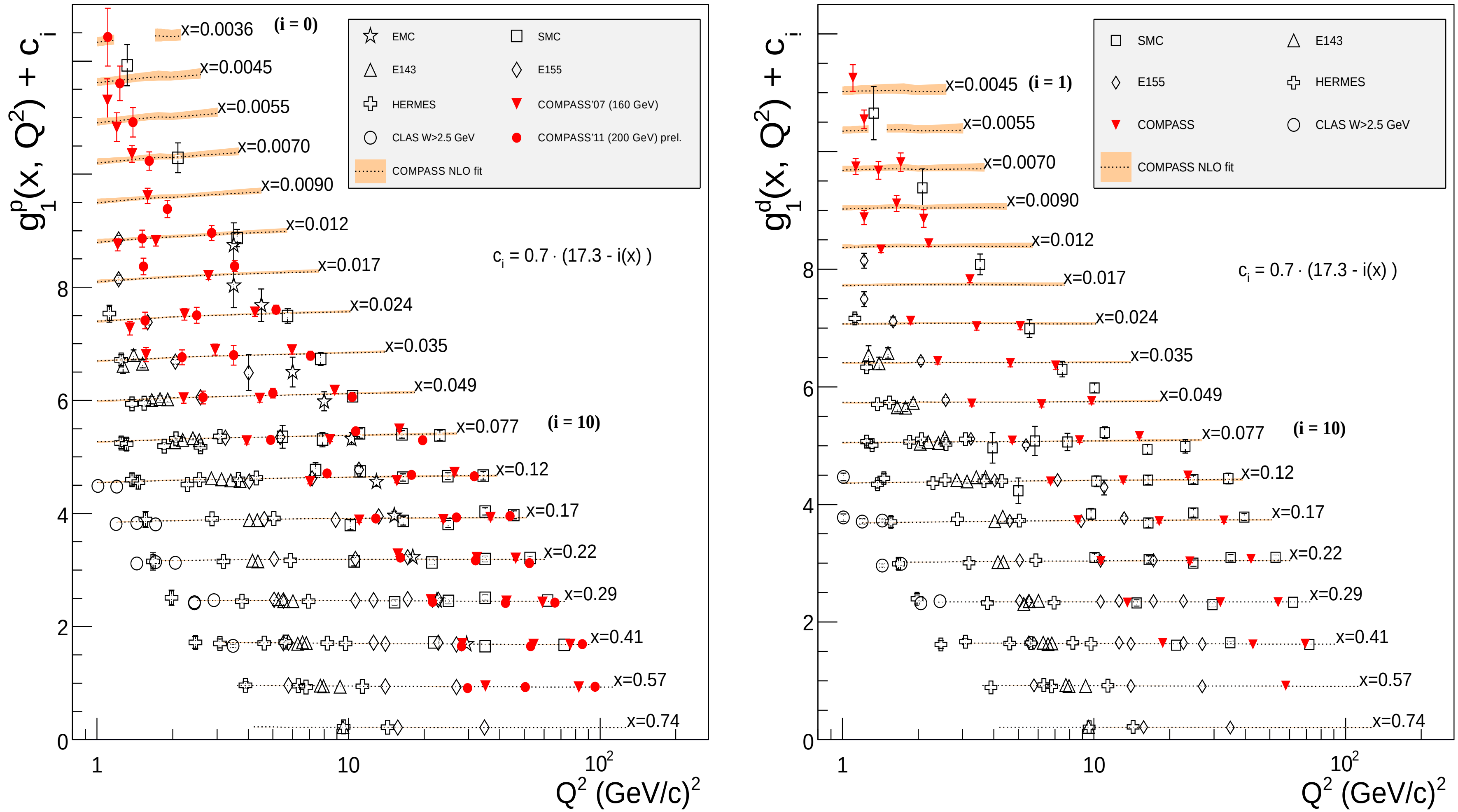
**for the gluon density $\Delta f_g$ and the flavour-singlet quark distribution**

$$\Delta f_{\mathrm{q}} = \sum_{i=1}^{n_f} \{\Delta f_{q_i} + \Delta f_{\bar{q}_i}\}$$

**Quark helicity-difference projector: $\not{p}\gamma_5$ – non-trivial in dim. reg.**

# 2014 world data on the structure function $g_1$

**M. Wilfert, DIS 2014 [courtesy of the Mainz COMPASS group]**




**Future, possibly: eRHIC** see Aschenauer et al, eRHIC Design Study, 09/2014

# Second-order calculations of the 1990s

Splitting functions $\Delta P_{ik}^{(1)}$, coefficient functions for $g_1$ in polarized e.m. DIS

- Structure function $g_1$ analogous to $F_{2,3,L}$ : $\Delta P_{\rm qq}^{(1)},\ \Delta P_{\rm qg}^{(1)},\ c_{g_1,\,\rm q/g}^{(2)}$

  Zijlstra, van Neerven (1993) [Err. ’97, 2007]

  $\gamma_5$: Larin scheme $\Leftrightarrow$ ’t Hooft, Veltman (’72); Breitenlohner, Maison (’77)

- All NLO splitting functions $\Delta P_{ij}^{(1)}$ using OPE / lightlike axial gauge

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**Transformation from L/HVBM to $\overline{\rm MS}$ scheme at NNLO** Matiounine et al. (1998)

$$Z_{ik}(\alpha_{\sf S}(\mu^2)) \;=\; \delta_{iq}\delta_{kq}\left(a_{\sf S}\, z_{\rm ns}^{(1)} + a_{\sf S}^2\,(z_{\rm ns}^{(2)} + z_{\rm ps}^{(2)}) + \;\ldots\right)$$

Non-singlet: $c_{g_1} \leftrightarrow c_{F_3}$. Pure singlet, $z_{\rm gq}^{(n)} = 0$: no second calculation yet

# Large-$x$ limits of the splitting functions

$x \to 1$ (threshold): expect suppression of helicity flip by $(1-x)^2 \leftrightarrow 1/N^2$

cf. Brodsky, Burkhardt, Schmid (1994)

E.g., leading-order (LO) splitting functions, with $\delta_{ik}^{(0)} \equiv P_{ik}^{(0)} - \Delta P_{ik}^{(0)}$

$$\delta_{\rm qq}^{(0)} = 0 \ , \quad \delta_{ik}^{(0)} = \text{const} \cdot (1-x)^2 + \ldots \quad \text{for} \quad ik = \rm qg,\ gq,\ gg$$

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NLO, in the standard version ('M') of $\overline{\rm MS}$

Mertig & van Neerven; Vogelsang

$$\delta_{ij}^{(1)} = \text{const} \cdot (1-x)^a \quad \text{for} \quad ik = \rm qq,\ gg\ (\mathit{a}=1),\ qg\ (\mathit{a}=2)$$

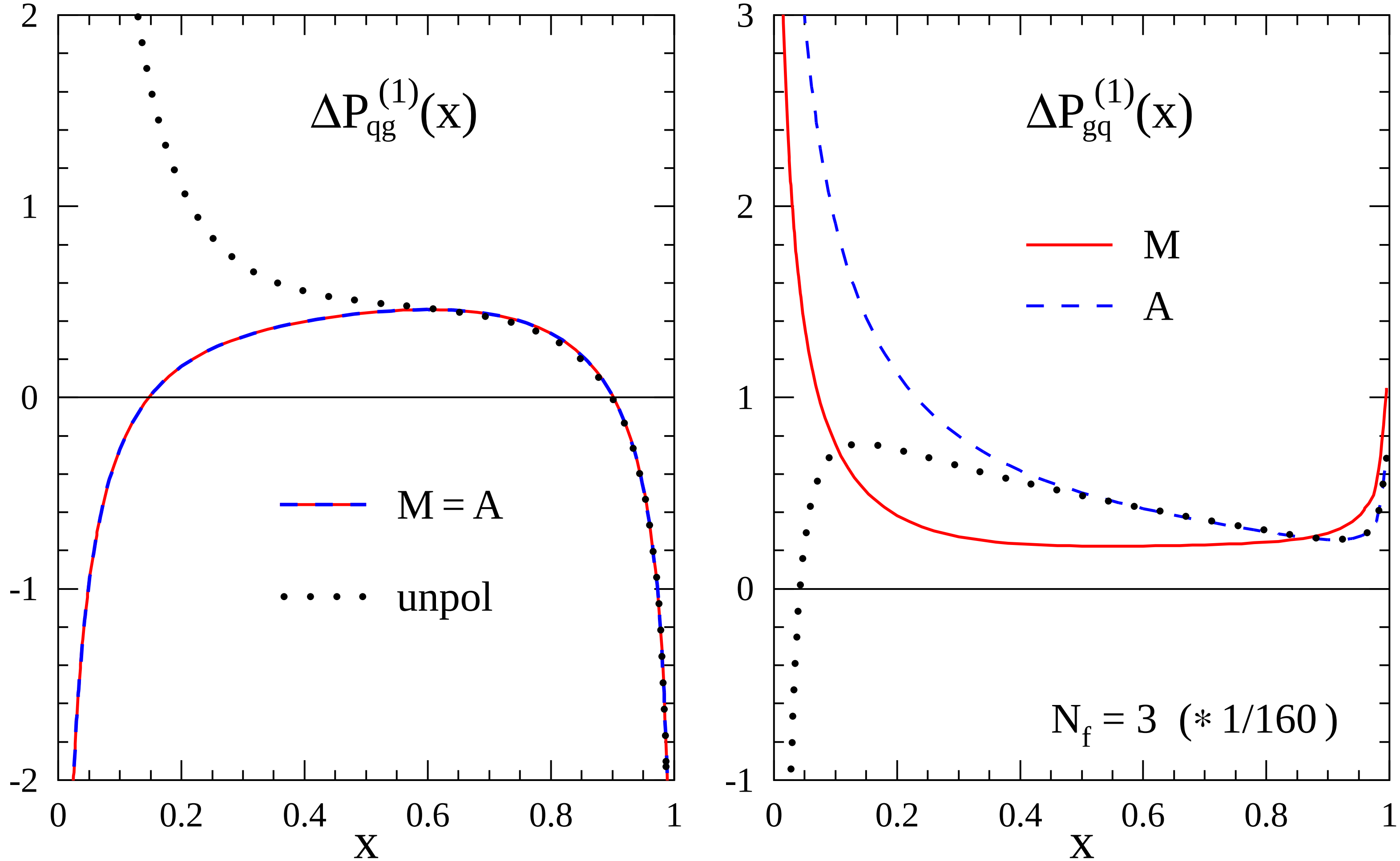
$$\begin{aligned}\delta_{\rm gq}^{(1)} &= 8C_F(C_A-C_F)(2-x)\ln(1-x) + 4C_F\beta_0 - 6C_F^2 \\ &\quad + (20/3C_FC_A + 2C_F^2 - 8/3C_Fn_f)(1-x) + \mathcal{O}(1-x)^2\end{aligned}$$

Physics or scheme artifact? Flavour-singlet physical kernels, if available for corresponding quantities, can provide insight

cf. Furmanski, Petronzio (1981)

$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2} f + CPf = \Big(\beta(a_{\rm s})\frac{dC}{da_{\rm s}} + CP\Big) C^{-1}F = KF$$

# Off-diagonal NLO splitting functions

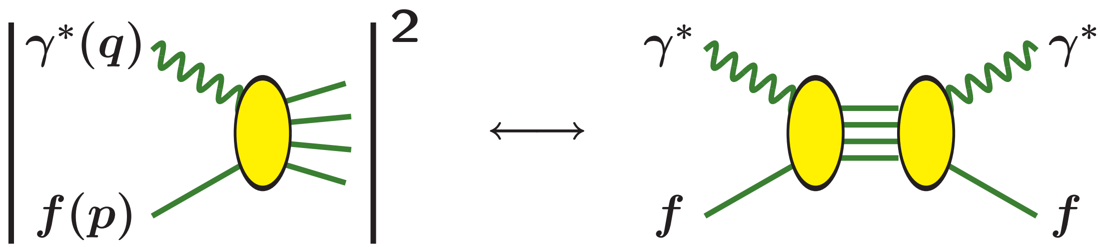



**M: standard scheme, A: additional $z_{\mathbf{gq}}^{(1)} = -\Delta P_{\mathbf{gq}}^{(0)}$ in trf. from Larin scheme,**

**removes all $(1-x)^0$ and $(1-x)^1$ terms in $\delta_{\mathbf{gq}}^{(1)}$ ...**

# Third order via forward Compton amplitudes

**Optical theorem: probe-parton total cross sections $\longleftrightarrow$ forward amplitudes**




**Dispersion relation in $x$ : coefficient of $(2p\cdot q)^N$ $\leftrightarrow$ $N$-th Mellin moment**

$$A^N \;=\; \int_0^1 dx\, x^{N-1} A(x)$$

**Unpol.: Larin, Nogueira, van Ritbergen, Vermaseren (1994) [Mincer], MVV (2004) [all-N]**

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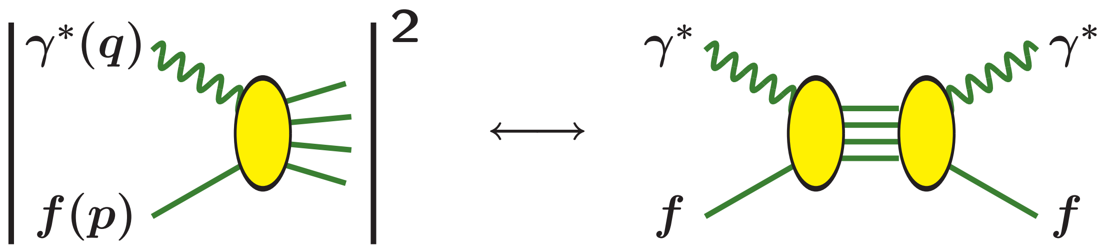


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**Pol. case: projection of partonic tensor on $g_1$ in $D = 4 - 2\varepsilon$ dimensions**

$$\hat{g}_1 \;=\; 2\,[(D-2)(D-3)(p\cdot q)]^{-1}\,\varepsilon_{\mu\nu pq}\,\widehat{W}_A^{\mu\nu}$$

$\varepsilon^{-1}$: $\Delta P_{\rm qq}^{(2)}(N)$, $\Delta P_{\rm qg}^{(2)}(N)$ **MVV (Loops & Legs 2008)**

$\epsilon^0$ : **N$^3$LO coefficient functions for $g_1$, mod. scheme transf. of pure singlet**

# Treatment of the forward-Compton integrals

Combine identities: integration by parts, scaling, Passarino-Veltman type

$\Rightarrow$ Difference equations for $I(N)$ [ recall: coefficient of $(2p\cdot q)^N$ ]

$$a_0(N)I(N) - \ldots - a_n(N)I(N-n) = I_0(N)$$

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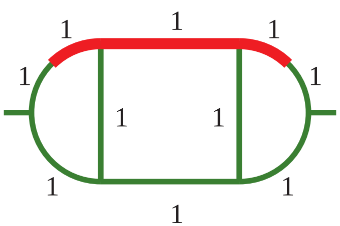
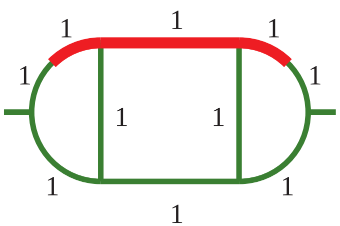
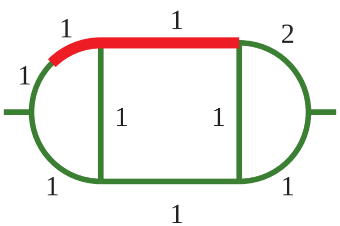
**Simple scalar example [ red line: flow of massless parton momentum $p$ ]**




$$+ \frac{N+3+3\epsilon}{N+2}\,\frac{2p\cdot q}{q^2}$$




$$= \frac{2}{N+2}$$




**Successive reduction to simpler (lower topologies or ‘less red’) integrals**

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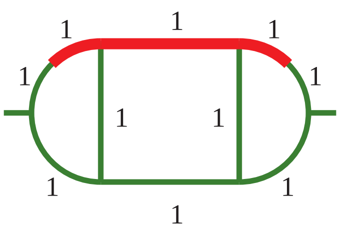


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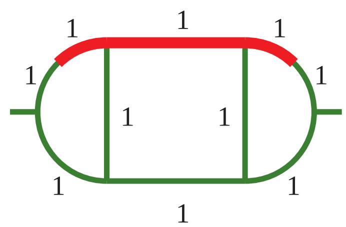


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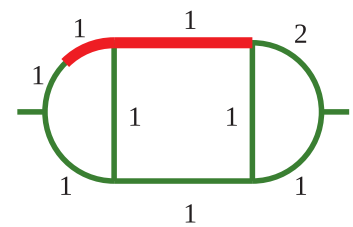


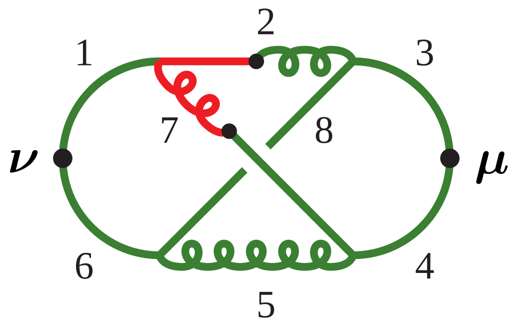

Successive reduction to simpler (lower topologies or 'less red') integrals

Essential: non-symbolic case for low $N$ can be done via Mincer

Gorishny, Larin, Tkachov (84, 89); Larin, Tkachov, Vermaseren (91)

Check of all-$N$ code and results at all stages: $I(N=2,3,4,\ldots) = ?$

# Numerators for a non-planar $\mathrm{NO}_{27}$ diagram




$$\sim \frac{\gamma(i_1, p_7, i_2, p_4, \mu, p_3, i_3, p_8, i_2, p_6, \nu, p_1, i_1, p+p_2, i_3)}{p_1^2 \dots p_8^2 \, (p+p_2)^2 (p+p_7)^2}$$

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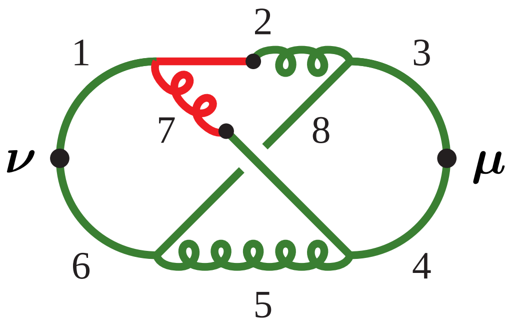


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**Unpolarized splitting functions:** $\cdot\, \gamma(p)\, \delta_{\mu\nu}$ **sufficient,**

**no** $(p_2^2)^{-1}$**, num.** $(p_2 \cdot p)^{k_2} (p_3 \cdot p)^{k_3} (p_2 \cdot q)^{k_9}$ **with** $k_2 + k_3 + k_9 \leq 3$

**Coefficient functions for** $F_2/F_L$**: need also** $\cdot\, \gamma(p)\, p_\mu p_\nu / (p \cdot q)^2$**,**

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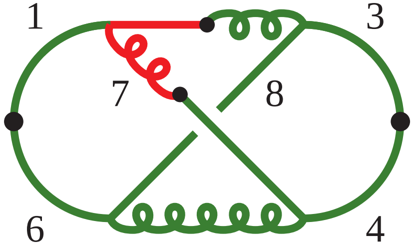


$$\nu \quad \mu \sim \frac{\gamma(i_1, p_7, i_2, p_4, \mu, p_3, i_3, p_8, i_2, p_6, \nu, p_1, i_1, p+p_2, i_3)}{p_1^2 \ldots p_8^2\, (p+p_2)^2 (p+p_7)^2}$$

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**Polarized splitting functions, structure function** $g_1$**:** $\cdot\, \gamma(p, 5)\, \varepsilon_{\mu\nu pq} / (p \cdot q)$,

**also** $(p_2^2)^{-1}$, **numerators** $(p_3 \cdot p)^{k_3} (p_2 \cdot q)^{k_9}$ **with** $k_3 + k_9 \leq 5$

**Helicity-difference projector: Larin scheme** $\not{p}\, \gamma_{5,L} = \frac{1}{6}\, \varepsilon_{p\mu\nu\rho}\, \gamma^\mu \gamma^\nu \gamma^\rho$

# One colour factor of $\Delta P_{qg}^{(2)}(N)$

$$
\begin{aligned}
\frac{1}{8}\Delta P_{\rm qg}^{(2)}(N)\Big|_{C_F^2 n_f} = {} & 2\Delta p_{\rm qg}(-S_{-4}+2S_{-2,-2}+4S_{1,-3}+2S_{1,1,1,1}-S_{1,1,2}-5S_{1,2,1} \\
& +4S_{1,3}+2S_{2,-2}-6S_{2,1,1}+6S_{2,2}+7S_{3,1}-3S_4) \\
& -3\zeta_3(2D_0^2+4D_1^2-9D_0+12D_1)+4S_{-3}(D_0^2-2D_0+2D_1)+8S_{1,-2}(2D_1^2-D_0+D_1) \\
& -2S_{2,1}(4D_0^2+2D_1^2-11D_0+11D_1)+S_{1,1,1}(5D_0^2-2D_1^2-21/2D_0+12D_1) \\
& -2S_{1,2}(2D_0^2-2D_1^2-5D_0+5D_1)+2S_3(3D_0^2+6D_1^2-11D_0+11D_1) \\
& +2S_{-2}(8D_1^3-5D_0^2-6D_1^2+10D_0-9D_1)-S_{1,1}(10D_0^3+6D_1^3-35/2D_0^2-5D_1^2 \\
& +29D_0-36D_1)+2S_2(4D_0^3+6D_1^3-10D_0^2-4D_1^2+17D_0-22D_1) -6D_2(S_{-2}+1) \\
& +S_1(7D_0^4+4D_1^4-43/2D_0^3-15D_1^3+99/2D_0^2+18D_1^2-78D_0+329/4D_1)+32D_1^5 \\
& -15/2D_0^4-3D_1^4+59/8D_0^3+53/4D_1^3+77/8D_0^2+213/8D_1^2-1357/32D_0+777/16D_1
\end{aligned}
$$

**All harmonic sums with argument $N$, $D_k = (N+k)^{-1}$, $\Delta p_{\rm qg} = 2D_1 - D_0$**

# One colour factor of $\Delta P^{(2)}_{\rm qg}(N)$

$$\frac{1}{8}\Delta P^{(2)}_{\rm qg}(N)\Big|_{C_F^2 n_f} = 2\Delta p_{\rm qg}(-S_{-4}+2S_{-2,-2}+4S_{1,-3}+2S_{1,1,1,1}-S_{1,1,2}-5S_{1,2,1} +4S_{1,3}+2S_{2,-2}-6S_{2,1,1}+6S_{2,2}+7S_{3,1}-3S_4)$$

$$-3\zeta_3(2D_0^2+4D_1^2-9D_0+12D_1)+4S_{-3}(D_0^2-2D_0+2D_1)+8S_{1,-2}(2D_1^2-D_0+D_1)$$
$$-2S_{2,1}(4D_0^2+2D_1^2-11D_0+11D_1)+S_{1,1,1}(5D_0^2-2D_1^2-21/2D_0+12D_1)$$
$$-2S_{1,2}(2D_0^2-2D_1^2-5D_0+5D_1)+2S_3(3D_0^2+6D_1^2-11D_0+11D_1)$$

$$+2S_{-2}(8D_1^3-5D_0^2-6D_1^2+10D_0-9D_1)-S_{1,1}(10D_0^3+6D_1^3-35/2D_0^2-5D_1^2$$
$$+29D_0-36D_1)+2S_2(4D_0^3+6D_1^3-10D_0^2-4D_1^2+17D_0-22D_1)\;-6D_2(S_{-2}+1)$$

$$+S_1(7D_0^4+4D_1^4-43/2D_0^3-15D_1^3+99/2D_0^2+18D_1^2-78D_0+329/4D_1)+32D_1^5$$
$$-15/2D_0^4-3D_1^4+59/8D_0^3+53/4D_1^3+77/8D_0^2+213/8D_1^2-1357/32D_0+777/16D_1$$

**All harmonic sums with argument $N$, $D_k=(N+k)^{-1}$, $\Delta p_{\rm qg}=2D_1-D_0$**

- **Weight-four sums: as in unpol. case, up to replacement $p_{\rm qg}\longrightarrow\Delta p_{\rm qg}$**
- **Very few terms with $D_2$, no corresponding primes in moment denom's**
- **No indices -1. Large-$N$ pol.-unpol. suppression separately for each sum**
- **$x\longrightarrow 0$ and $x\longrightarrow 1$ knowledge: $D^5_{0,1}$, $D^4_1$ and $S_{1,1,1}$ terms predictable**

# Accessing the lower row, $\Delta P^{(2)}_{\rm gq}$ and $\Delta P^{(2)}_{\rm gg}$

**$\Delta P^{(2)}_{\rm gq,\,gg}$ enter $\gamma^* f$ amplitudes only at order $\alpha_{\rm s}^4$: need direct gluon coupling**

**Unpol.: $F_2^{\rm e.m.}$ complemented by scalar $\phi$ with $\phi G^{\mu\nu} G_{\mu\nu}$ coupling to gluons**

**$\Leftrightarrow$ Higgs-exchange DIS in heavy-top limit** **Furmanski, Petronzio (1981)**

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Polarized case: non-(pseudo)scalar probe required

- Extend to supersymmetric case, as done for NNLO antenna functions

  Gehrmann-de Ridder, Gehrmann, Glover (2005)

- Consider graviton-exchange DIS

  Lam, Li (1981), cf. Stirling, Vryonidou (2011)

  Structure functions $H_k$, $k = 1-4, 6$: unpol. & pol. analogues of $(F_2, F_\phi)$

Drawback: lots of higher tensor integrals, far beyond 2004 calculation of $F_2$, $F_\phi, \ldots$ and 2008 extension to $g_1$ $\Rightarrow$ fall back to fixed-$N$ Mincer calculation

# Improved diagram handling and Mincer code

"The problem was that Andreas needed a few more moments to produce nice physics"

Jos, 'Xtreme Manipulations', Loops & Legs 2014

- **Combine diagrams of same subtopology, colour factor & flavour class**

  **5176 $\longrightarrow$ 1142 quark and 15208 $\longrightarrow$ 1249 gluon 'diagrams', no loss of information**

- **Optimize flow of parton momentum $P$ through the diagrams**

  **Minimal number of $P$-propagators, if same: some routes 'more equal than others'**

- **(Experimentally) improve number of steps per FORM module**

  **Reduce slowing down by either generating too many terms or sorting too often**

- **Render vector and scalar-product substitutions more economical**

  **Use binomial coefficients, removal of tadpoles, 'slow substitution' of high powers**

- **Increase size of tables for the non-planar (NO) integrals**

  **Enhanced $\{n_2, n_5, n_7, n_8\}$ tabulated for $n_2 + n_5 + n_7 + n_8 \leq 31$ (was 12)**

# Mincer moments of $\Delta P^{(2)}_{\mathrm{gq}}$, coeff's of $C_F^3$

**Odd moments $N \geq 3$ are accessible** **Lam, Li (1981)**

**Results of the Mincer calculation, coefficient of $C_F^3$, Larin scheme**

**N = 3:** $186505/(3^5 2^5)$

**N = 5:** $9473569/(5^5 3^5 2^2)$

**N = 7:** $-509428539731/(7^5 5^4 3^2 2^{11})$

**N = 9:** $-266884720969207/(7^4 5^5 3^{10} 2^7)$

**N = 11:** $-334956658917082965l/(11^5 7^4 5^4 3^9 2^7)$

**N = 13:** $-751774767290148022507/(13^5 11^4 7^3 5^3 3^7 2^8)$

**N = 15:** $-23366819019913026454180147/(13^4 11^4 7^4 5^5 3^9 2^{16})$

**N = 17:** $-305214227818628090680174170947/(17^5 13^4 11^4 7^4 5^4 3^{10} 2^{10})$

**N = 19:** $-570679648684656807578199791973487/(19^5 17^4 13^4 11^4 7^3 5^5 3^7 2^9)$

**N = 21:** $-2044304092089235762279148843319979/(19^4 17^4 13^4 11^4 7^5 5^3 3^9 2^{11})$

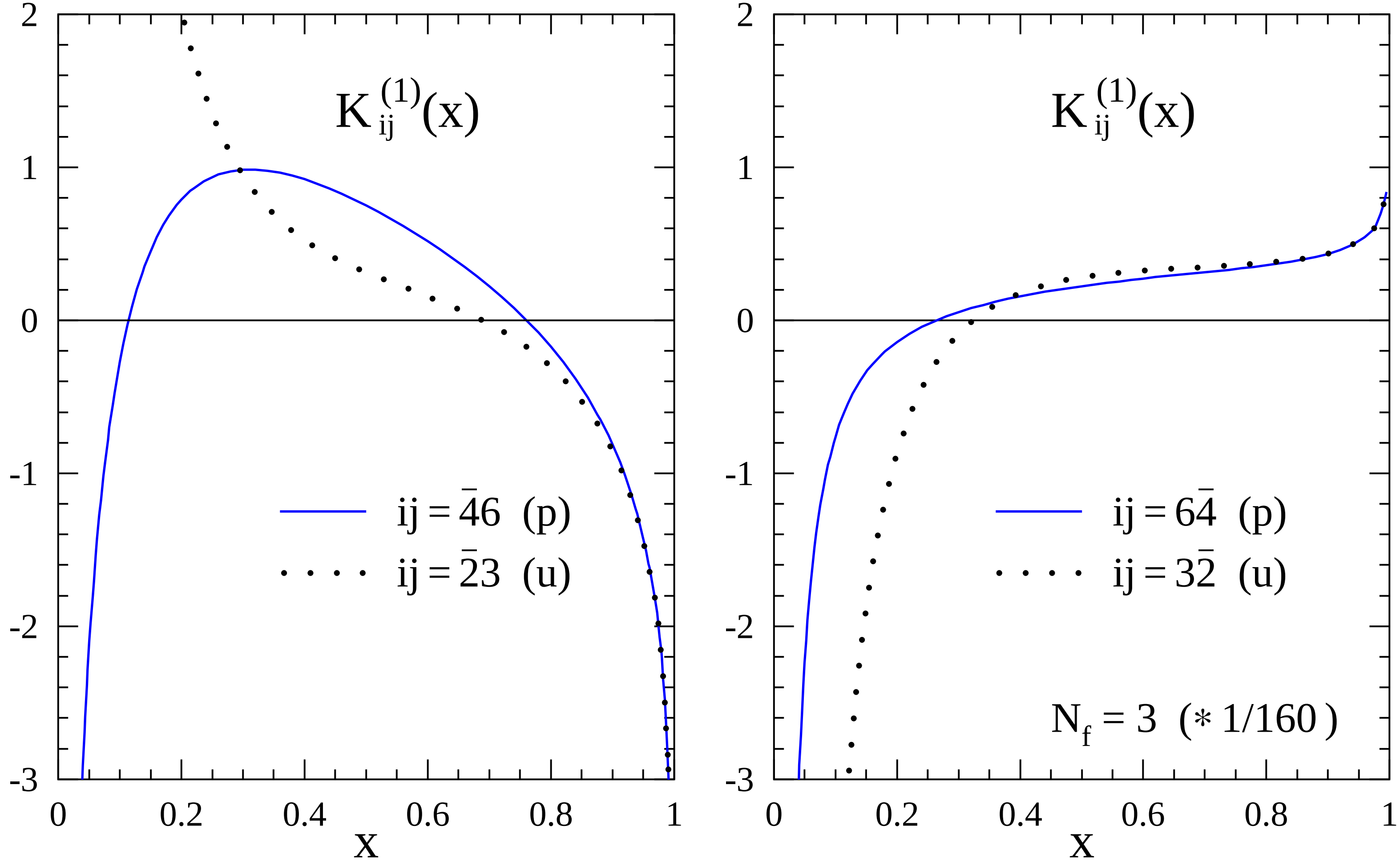
**N = 23:** $-289119840113761409530260333250139823739/(23^5 19^4 17^4 13^4 11^4 7^4 5\, 3^9 2^{13})$

**N = 25:** $-1890473255283802937678830745102921869938637/(23^4 19^4 17^4 13^5 11^4 7^4 5^{10} 3^5 2^{12})$

**Machines: Zeuthen, NIKHEF (hardest cases), ulgqcd cluster Liverpool (bulk production)**

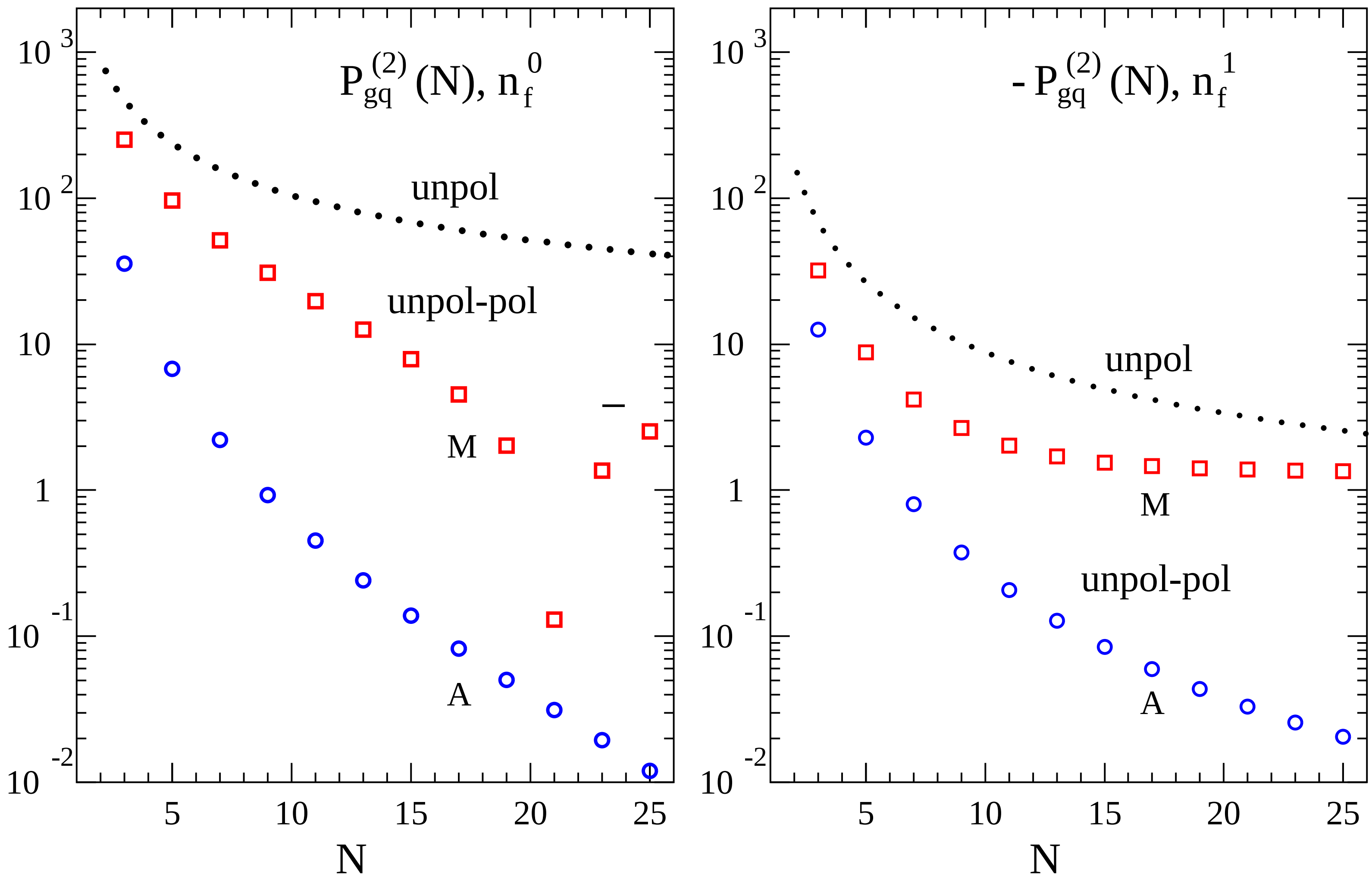
# NLO physical kernels for graviton exchange

**Unpol.: structure funct's $H_{\bar{2}}$ (LO: q) and $H_3$ (LO: g). Pol. analogues: $H_{\bar{4}}$, $H_6$**




$\Rightarrow$ **Large-$x$ behaviour of standard $\Delta P^{(1)}_{gq}$ is a factorization-scheme artifact**

# Large-$N$ (non-)suppression of $\Delta P^{(2)}_{\rm gq}$




Consistent with $\frac{1}{N^2}$ suppressed difference in $A$-scheme, $z^{(2)}_{\rm gq} = -\frac{1}{2}\Delta P^{(1)}_{\rm gq}$

# Determination of $\Delta P_{\rm gq}^{(2)}$ at all $N$

Critical: $n_f^0$ parts. Coefficient of weight-4 sums fixed from unpolarized case

Weight $\leq$ 3 : 2$\times$32 coefficients with $D_0$ or $D_1$, plus up to 11 sums with $D_{-1}$

- 2$\times 12$ coefficients (of $D_0^1$ & $D_1^1$) fixed by $1/N^2$ $A$-scheme suppression
- 3 + 3 coefficient fixed by small-$x$ & large-$x$ (i.e., $S_{1,1,1}$) knowledge

$\Rightarrow$ Up to 45 unknown integer coefficients vs 12 odd moments $3 \leq N \leq 25$

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In-house primes program (Jos): analysis of prime decomposition, derivation of relations ( $\lesssim 10$) between coefficients via the Chinese remainder theorem

Eliminate four to six 'unpleasant' coefficients, e.g., $D_0^2$, $D_1^2$, $D_0^2 S_1$, $D_1^2 S_1$

Turn to the number-theory professionals (LLL algorithm),   cf. Velizhanin (12)

www.numbertheory.org/php/axb.html   (Keith Matthews, Queensland)

'Solves a system of linear Diophantine equations . . . via the Havas-Majewski-Matthews LLL-based algorithm. . . . .  We find . . . the solutions X with minimal length, using a modification of the Fincke-Pohst algorithm'

# One colour factor of $\Delta P^{(2)}_{\rm gq}(N)$

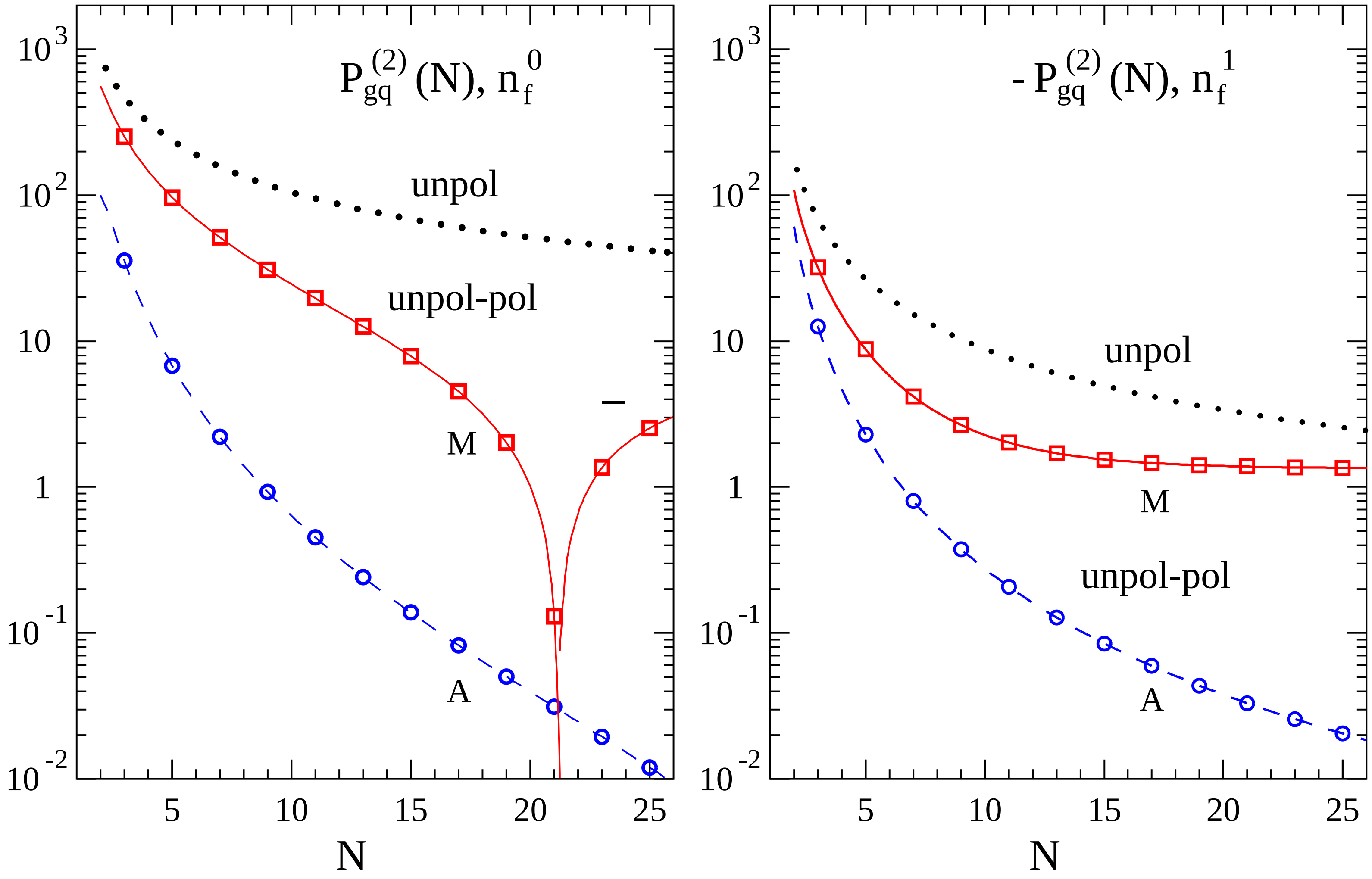
$$
\begin{aligned}
\frac{1}{8}\Delta P^{(2)}_{\rm gq}(N)\Big|_{C_F^3} \;=\;& 2\Delta p_{\rm qg}(-S_{-4}+6S_{-2,-2}+4S_{1,-3}+2S_{1,1,1,1}+S_{1,1,2}\\
&+3S_{1,2,1}-3S_{1,3}+2S_{2,-2}+2S_{2,1,1}-2S_{2,2})\\
&+6\zeta_3\Delta p_{\rm qg}(2S_1-3)-4S_{-3}(2D_0^2-D_0+D_1)-8S_{1,-2}(D_1^2-2D_0+2D_1)\\
&+S_{1,1,1}(2D_0^2-5D_1^2-6D_0-3/2D_1)-2S_{1,2}(D_1^2+4D_0-D_1)\\
&-S_{2,1}(4D_0^2+4D_1^2-4D_0+7D_1)+S_3(2D_0^2+D_1^2+6D_0-3/2D_1)\\
&-S_{-2}(8D_1^3+4D_0^2+18D_1^2-26D_0+24D_1)+2S_2(D_1^3+2D_1^2+10D_0-4D_1)\\
&-S_{1,1}(6D_0^3+6D_1^3+4D_0^2+5D_1^2+2D_0-7/4D_1)\;-\;6D_{-1}(S_{-2}+1)\\
&-S_1(6D_0^4+7D_1^4+4D_0^3+23/2D_1^3-27/2D_0^2+39/4D_1^2-8D_0+23/4D_1)\\
&-8D_0^5-12D_1^5+23D_0^4-28D_1^4-39/4D_0^3-427/8D_1^3-341/8D_0^2-767/8D_1^2\\
&+2427/16D_0-4547/32D_1
\end{aligned}
$$

**All harmonic sums with argument $N$, $D_k=(N+k)^{-1}$, $\Delta p_{\rm gq}=2D_0-D_1$**

**$C_FC_A^2$, $C_F^2C_A$ parts somewhat longer, rest much simpler ($N=25$ not needed)**

**All-$N$ formula for $\Delta P^{(2)}_{\rm gg}(N)$ analogous; overall most difficult: its $C_A^3$ part**

# All-$N$ result for $n_f^0$ anf $n_f^1$ parts of $\Delta P_{\mathrm{gq}}^{(2)}$




Difference $\delta_{\mathrm{gq}}^{(2)} = P_{\mathrm{gq}}^{(2)} - \Delta P_{\mathrm{gq,A}}^{(2)}$ indeed analytically suppressed by $\frac{1}{N^2}$

# Higher-$N$ checks, first-moment results

Most difficult colour factors of both cases: all moments to $N = 25$ used for determining the coefficients $\Rightarrow$ validate results by computing $N = 27,\ 29$

$$-\Delta P^{(2)}_{\rm gq}(27) = 460977038358760543281329153084972633526481072 7/ \left(23^4\, 19^4\, 17^4\, 13^4\, 11^4\, 7^5\, 5^8\, 3^{15}\, 2^{13}\right) C_F^3\ +\ \ldots$$

Total execution time: 256 874 306.6 sec. Maximum disk space: 1 261 024 031 636 bytes

Plus Mincer check of $\Delta P^{(2)}_{\rm gq}(29)$ for $C_A - 2C_F \rightarrow 0$ and $\Delta P^{(2)}_{\rm gq}(27, 29)|_{C_A^3}$

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First moments: transform to $x$-space in terms of HPLs, then calculate $N = 1$

Remiddi, Vermaseren (99)

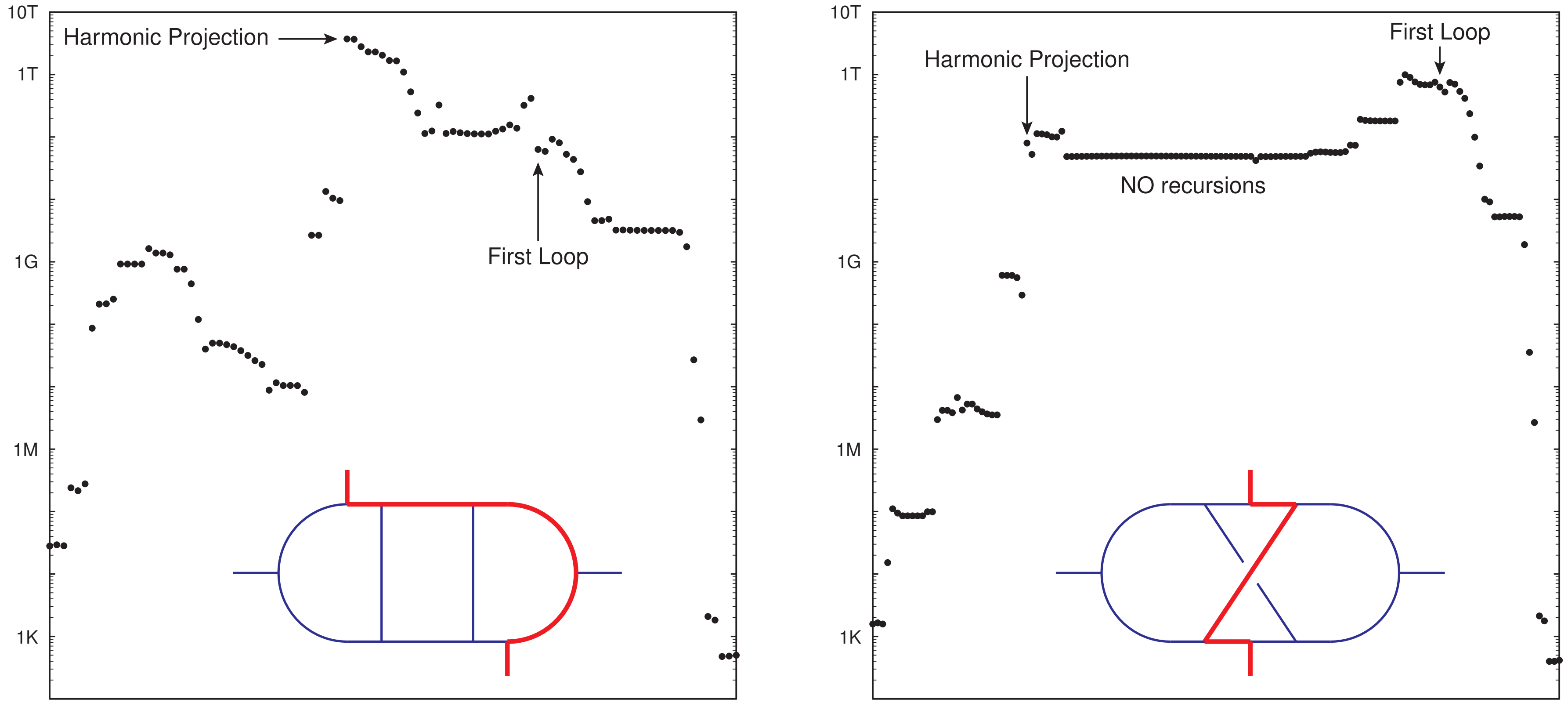
$$\Delta P^{(2)}_{\rm gq}(N\!=\!1) \;=\; \frac{1607}{12}\, C_F C_A^2 \;-\; \frac{461}{4}\, C_F^2 C_A \;+\; \frac{63}{2}\, C_F^3 \\ +\left(\frac{41}{3} - 72\zeta_3\right) C_F C_A n_f \;-\; \left(\frac{107}{2} - 72\zeta_3\right) C_F^2 n_f \;-\; \frac{13}{3}\, C_F n_f^2$$

$$\Delta P^{(2)}_{\rm gg}(N\!=\!1) \;=\; \frac{2857}{54}\, C_A^3 \;-\; \frac{1415}{54}\, C_A^2 n_f \;+\; \ldots \;=\; \beta_2$$

– another check

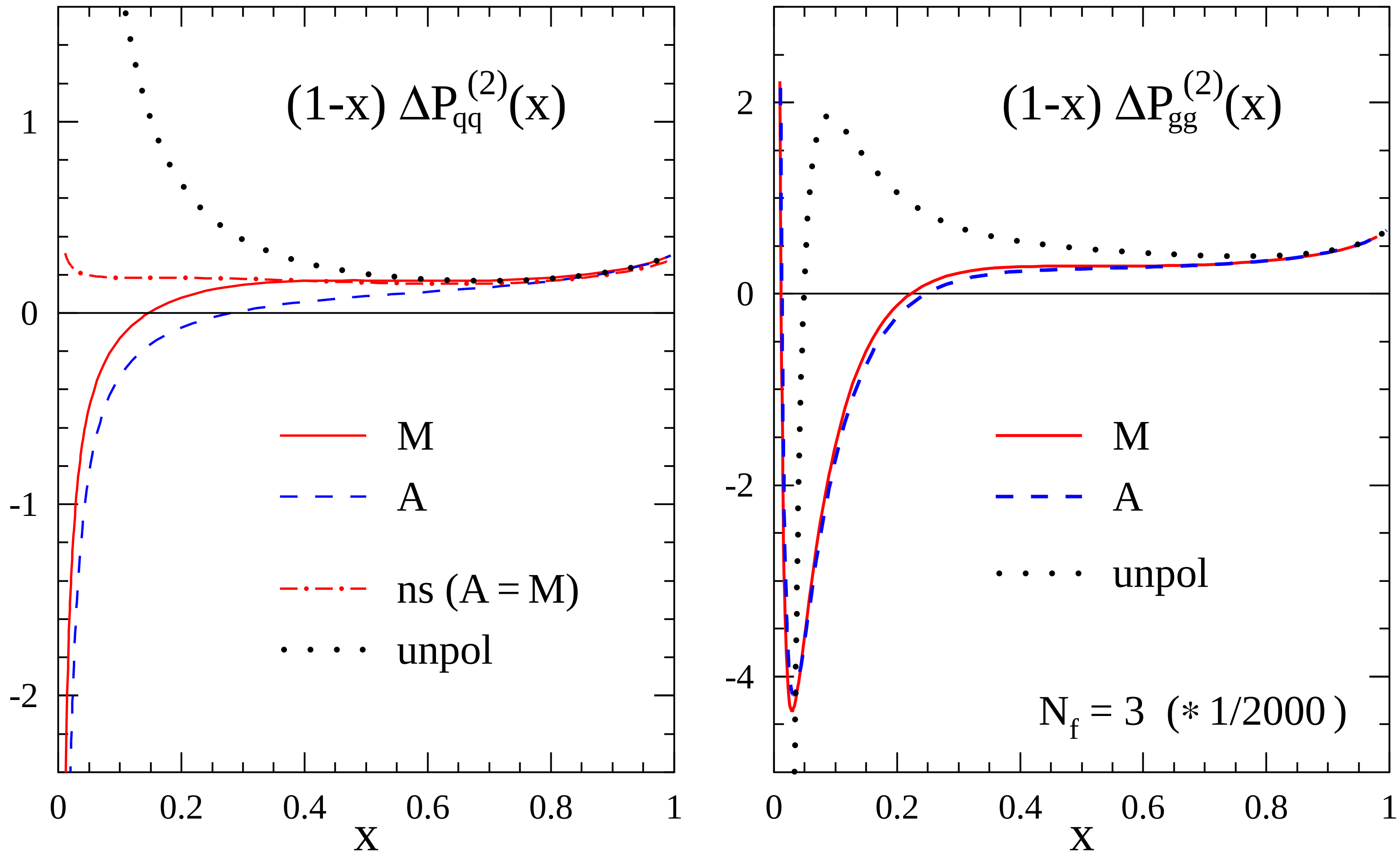
# Two of the hardest diagrams for Mincer

**$LA_{14}$ @ $N=29$, $NO_{25}$ @ $N=27$: expression sizes at the end of each module**



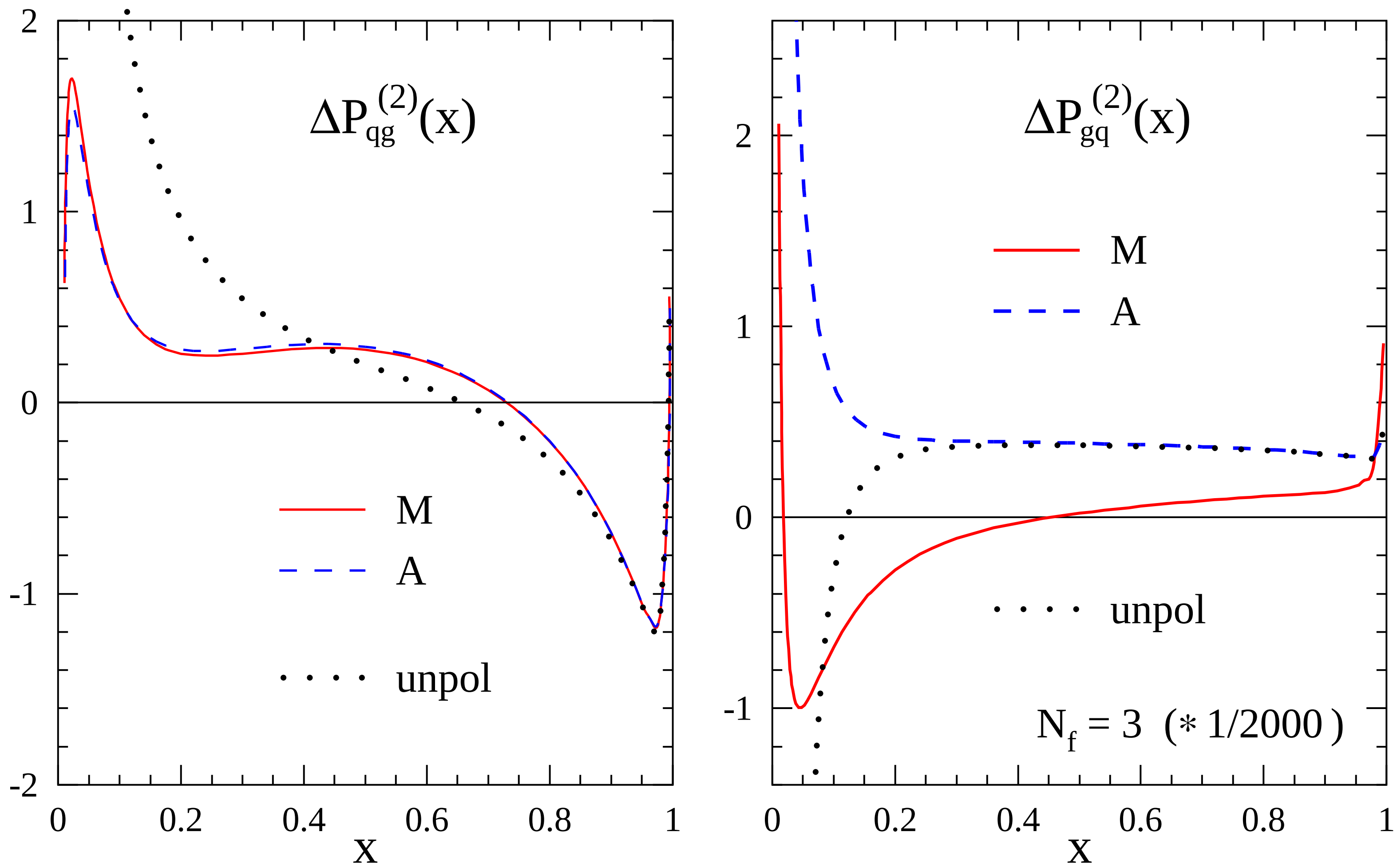

**Left: largest calculated diagram: about $10^7$ CPU seconds, 6.7 TB disk space for projection on $N$ (includes gzipped files of TFORM workers)**

# The diagonal splitting functions $\Delta P^{(2)}_{\mathrm{qq,\,gg}}(x)$




$\Delta P^{(2)}_{\mathrm{qq}} = \Delta P^{+(2)}_{\mathrm{ns}} + \Delta P^{(2)}_{\mathrm{ps}}$ **with** $\Delta P^{+(2)}_{\mathrm{ns}} = P^{-(2)}_{\mathrm{ns}}$ **as calculated in 2004**

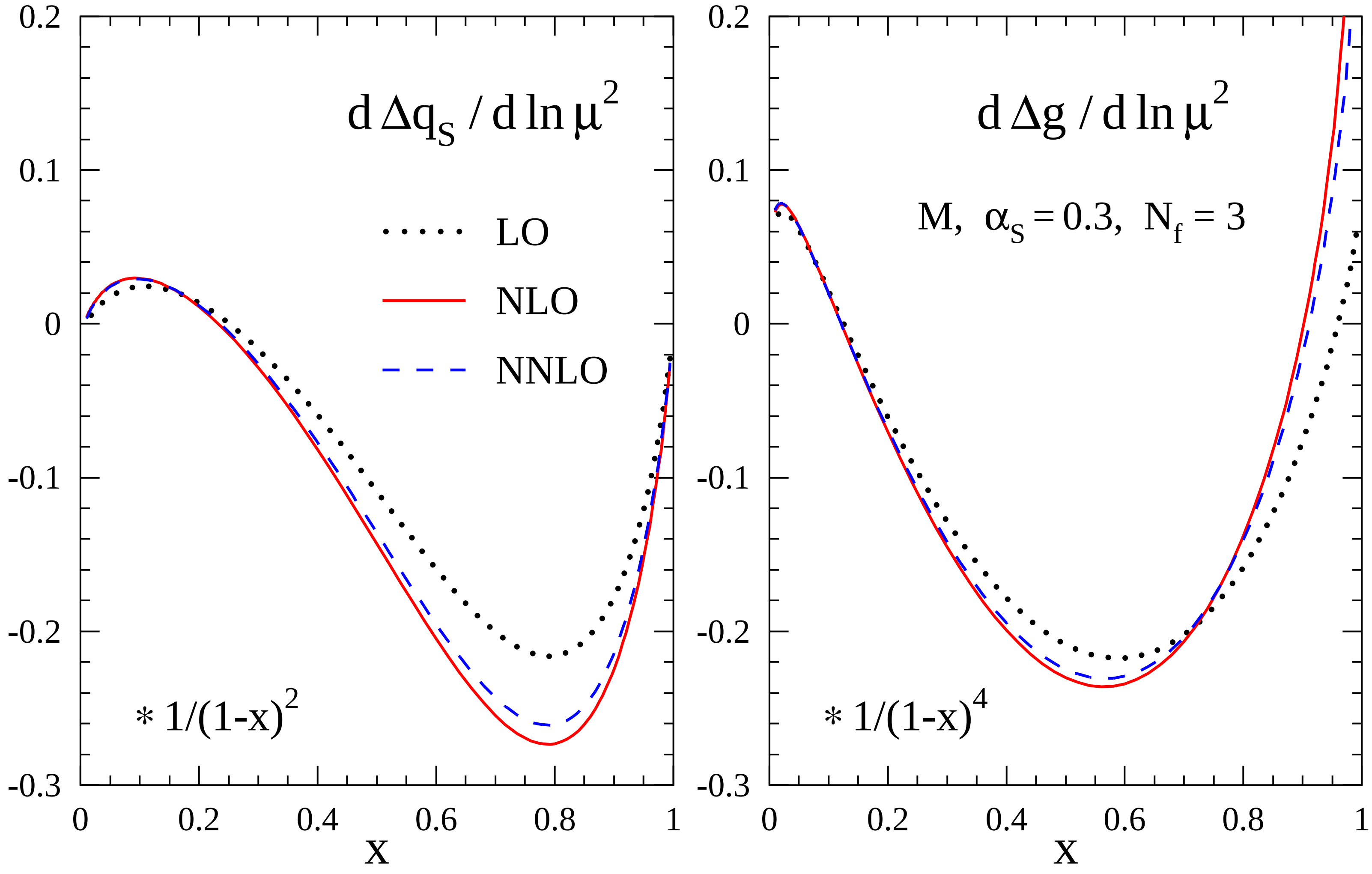
# The off-diagonal splitting functions $\Delta P^{(2)}_{\rm qg,\,gq}$




**AM difference of $\Delta P^{(2)}_{\rm qg}$:** $\quad z^{(2)}_{{\rm ps},\,A}(x) \;=\; z^{(2)}_{{\rm ps},\,M}(x) + 12\, C_F n_f (1-x)$

**which ensures** $\quad \Delta P^{(n)}_{{\rm ps},\,A}(N=1) \;=\; -2 n_f \Delta P^{(n-1)}_{{\rm gq},\,A}(N=1)$ $\quad$ **at** $n = 2$.

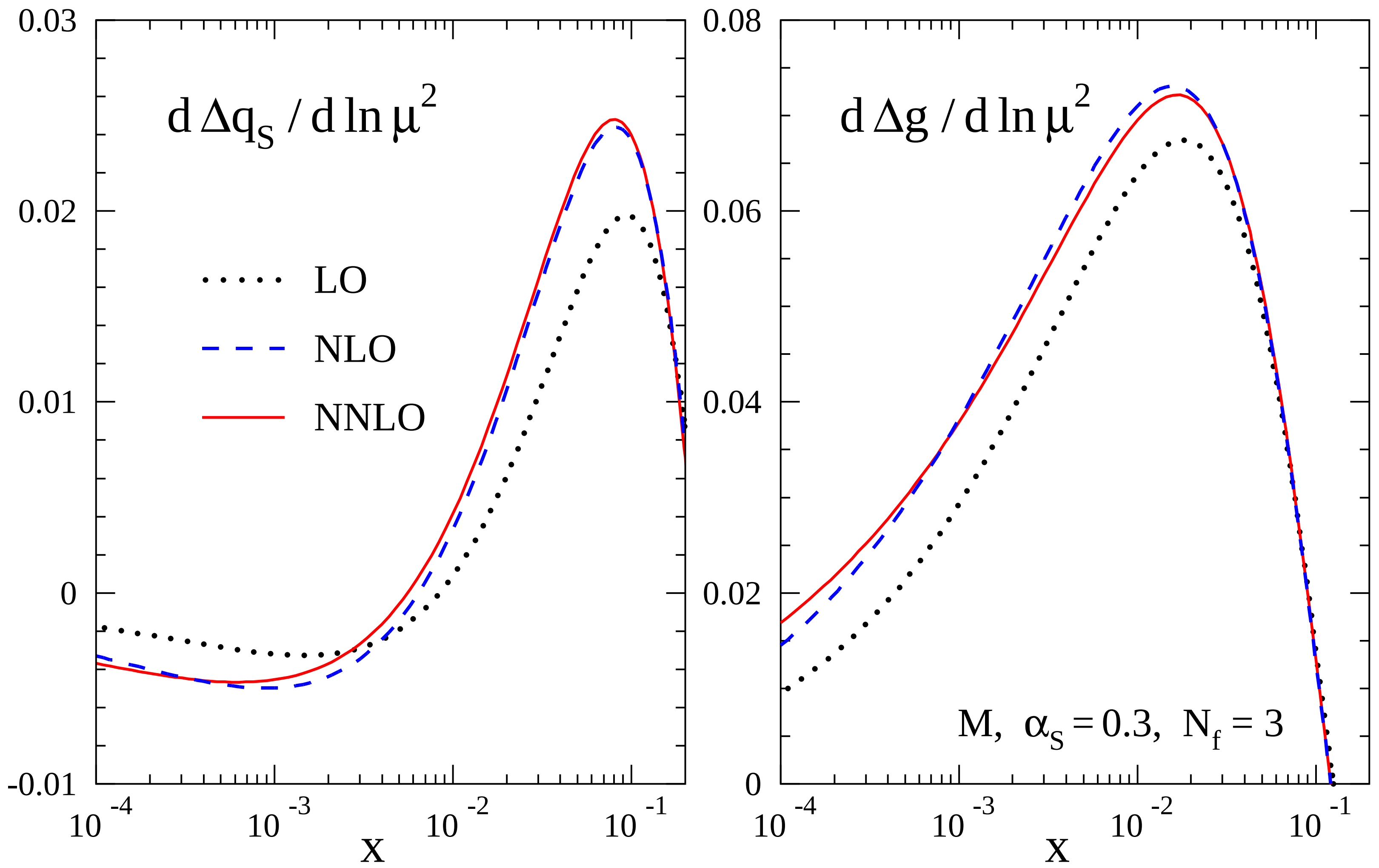
# Polarized quark and gluon evolution, large $x$




**Initial q and g distributions of QCD-Pegasus manual, evolution benchmarks**

**A.V. (2004); Salam, A.V. (HERA/LHC workshop 2004/5)**

# Polarized quark and gluon evolution, small $x$




**NNLO corrections fairly small down to small $x$ after convolution with input**

# Summary and outlook

‘Spät kommt Ihr – doch Ihr kommt. Der weite Weg, . . . , entschuldigt Euer Säumen.’

‘Late you come, yet you come. The long way, . . . , excuses your tarrying’

Schiller, Wallenstein (1799)

NNLO spin splitting functions $\Delta P_{ij}^{(2)}(x)$ calculated, finally

- New part (lower matrix row) by brute force, insight and number theory

  $3^{\mathrm{rd}}$-order Mincer calculation of gravition-exchange DIS also performed
  for upper row and unpolarized case: full agreement with previous results

- Agreement with all previous partial results (if interpreted properly) and
  expectations: $x \to 0, 1$, gg @ $N = 1$, leading $n_f$ Bennett, Gracey (1998)

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- Agreement with all previous partial results (if interpreted properly) and expectations: $x \rightarrow 0, 1$, gg @ $N = 1$, leading $n_f$ Bennett, Gracey (1998)

- Numerical effects small down to low $x$, as for unpolarized case

  Standard pol. $\overline{\text{MS}}$ factorization ($\gamma_5$ treatment) a bit unphysical for $x \rightarrow 1$
  – not a practical problem: no reason to change scheme after 19 years

- Re-calculation of transformation from Larin scheme $z_{\mathrm{iq}}^{(2)}$ worthwhile

  Knowledge of $z_{\mathrm{ps}}^{(3)}$ would fix $\mathrm{N^3LO}$ quark coefficient function for $g_1$